IMPERIAL

Control strategies for the McKean-Vlasov equation

How can we control long-term behaviour of many interacting agent systems?

Lucas M. Moschen

Supervised by Greg Pavliotis and Dante Kalise

Department of Mathematics, Imperial College London

Interacting agents

How do pairwise interactions create complex behaviour?

Opinion dynamics: simple models of how individual beliefs evolve under peer influence, external forces, and randomness.

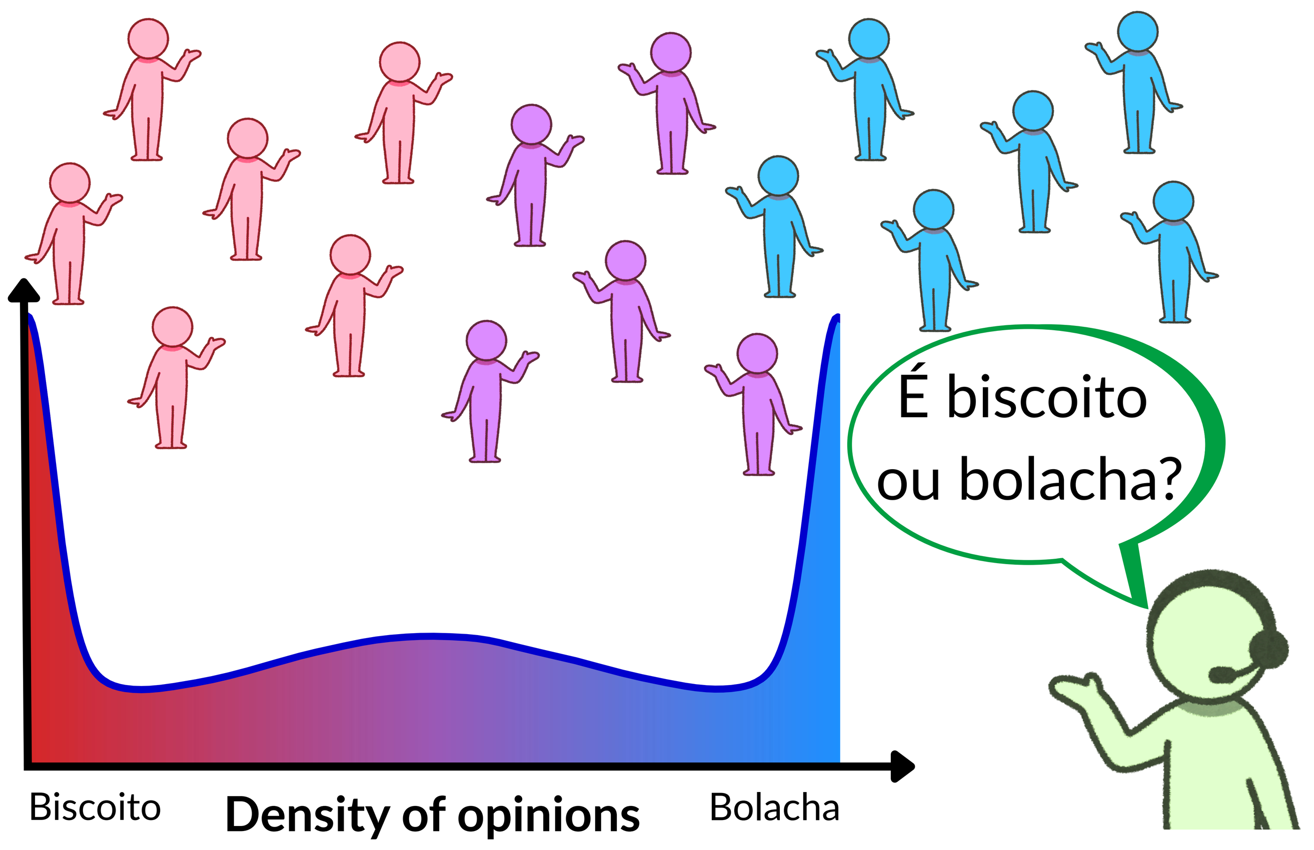

Figure 1: "Biscoito or bolacha?": a toy model of opinion dynamics, where people split into camps and the density shows their collective state.

- ✓ **Individual model:** Agents follow trajectories on domain Ω, influenced by a potential V, their **peers** via W, and noise with strength $\sigma > 0$:

$$dX_t^i = -\left(\nabla V(X_t^i) + \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_t^i - X_t^j)\right)dt + \sqrt{2\sigma}\, dB_t^i,$$

- ✓ **Large population:** When $N \to \infty$, the system can be described by a single **density** μ, satisfying the **McKean-Vlasov** (MV) equation:

$$\partial_t \mu = \sigma\Delta\mu + \nabla\cdot(\mu(\nabla V + \nabla W * \mu)), \quad (W * \mu)(x) = \int_\Omega W(x - x')\mu(x')dx',$$

- ✓ **Collective effects:** Interactions can cause the system to converge **very slowly** or to settle into **several different possible equilibria**.

Stabilisation: Selecting Equilibria

- ✓ **Multiple equilibria:** we can **choose** which outcome dominates.
- ✓ **Stabilisation:** modify the dynamics so the **desired equilibrium** becomes attractive.
- ✓ **Toy example:** two equilibria, $(1, 1)$ stable, $(0, 0)$ unstable. A simple **feedback control** can stabilise $(0, 0)$ instead.

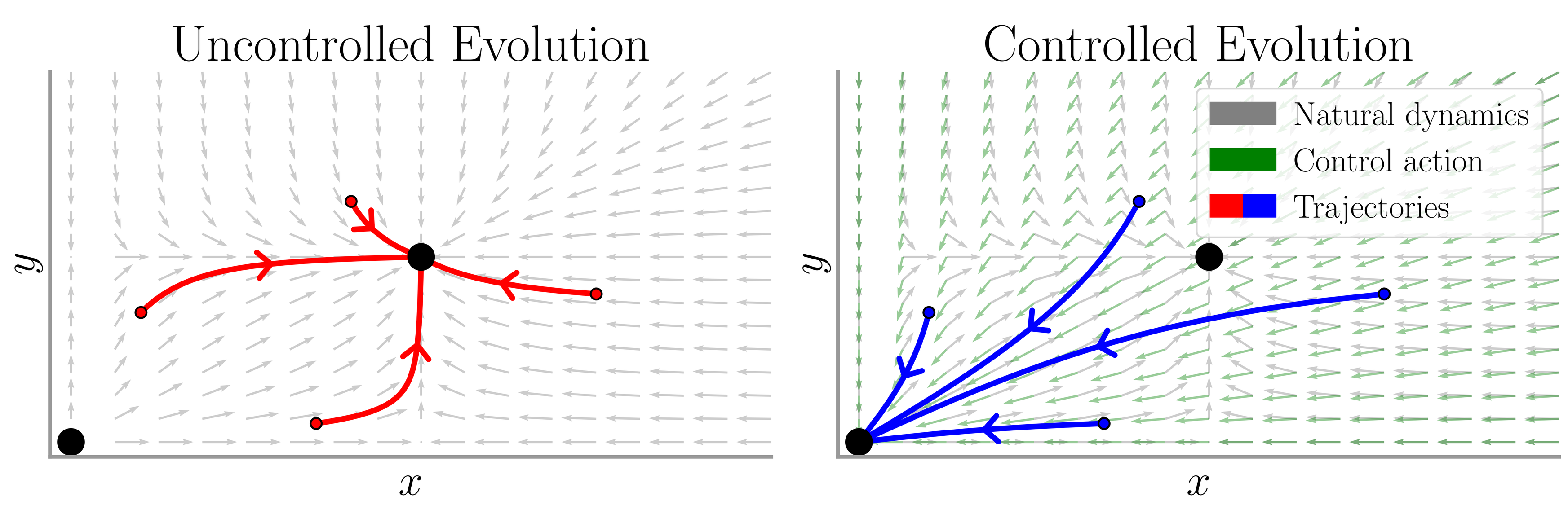

Figure 2: Control changes stability: without control, trajectories converge to $(1, 1)$; with control, they converge to $(0, 0)$.

Model: $\dot{z} = f(z) + Bu$ with equilibrium $\bar{z}$.

Goal: find a feedback control $u = -g(z)$ that yields ($C > 0$, $\delta > 0$)

$$\|z(t) - \bar{z}\| \leq Ce^{-\delta t}\|z(0) - \bar{z}\|.$$

Method: linearise f at $\bar{z}$ and analyse $A := Df(\bar{z})$. Then apply controls such as $u(t) = -K(z(t) - \bar{z})$ for a suitable K.

Extension to densities: replace z by the μ, linearise the MV equation, and use a suitable linear feedback.

Control Framework

How can we steer (or accelerate) the dynamics to $\bar{\mu}$?

We modify the potential V with **shape functions** $\{\alpha_j(x)\}_{j=1}^m$ and time controls $u_j(t)$:

$$V(x) + \sum_{j=1}^{m} u_j(t)\alpha_j(x).$$

Goal: exponential convergence $\|\mu(t) - \bar{\mu}\|_{L^2(\Omega)} \lesssim e^{-\delta t}$ for a given $\delta > 0$.

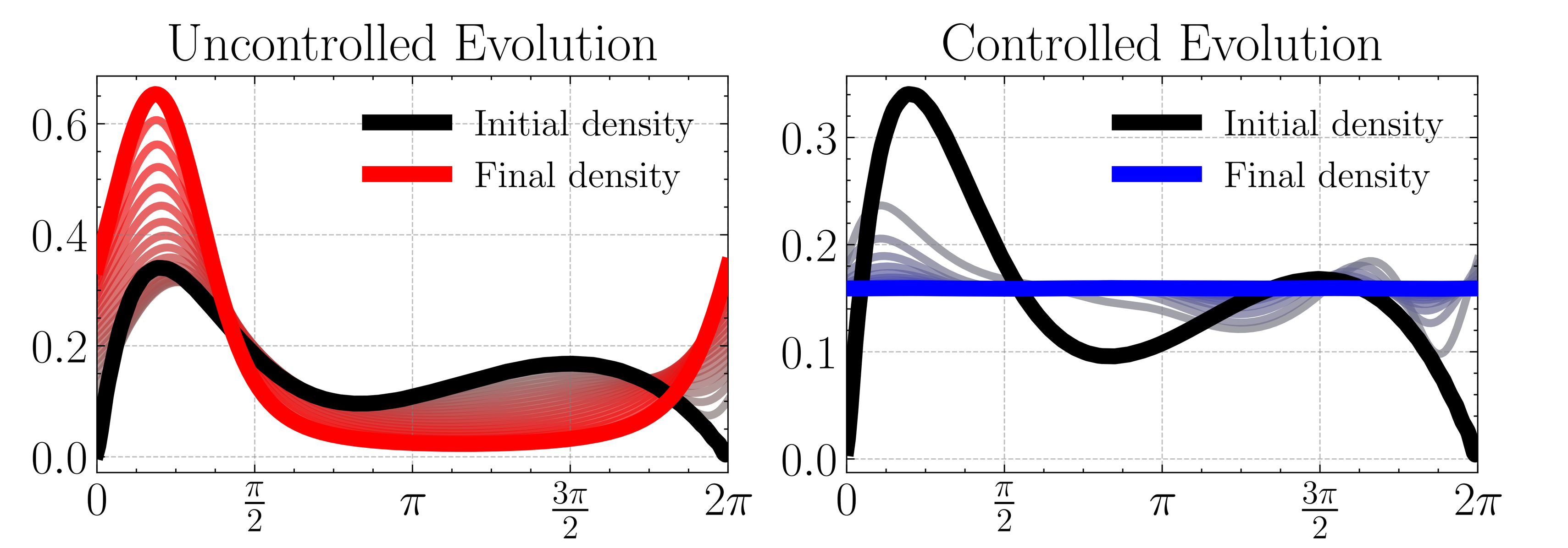

Figure 3: Kuramoto model ($W(x) = -5\cos(x)$, $V(x) = 0$, $\Omega = [0, 2\pi)$). Initial state (black) converges to red (uncontrolled) or blue (controlled) equilibria.

Method: linearise the MV equation at $\bar{\mu}$, set $u(t) = -\mathcal{K}(\mu(t) - \bar{\mu})$, and choose $\mathcal{K}$ solving an **algebraic Riccati equation** so that all eigenvalues satisfy $\Re\lambda \leq -\delta$, i.e., exponential decay at rate δ.

Nonlinear extension proof: this control stabilises the MV dynamics by bounding the **nonlinear terms** and applying a **fixed-point** argument.

Cosine-coupled rotator model: numerics

The feedback control drives the system to the desired equilibrium **exponentially**, in contrast to the slower uncontrolled dynamic.

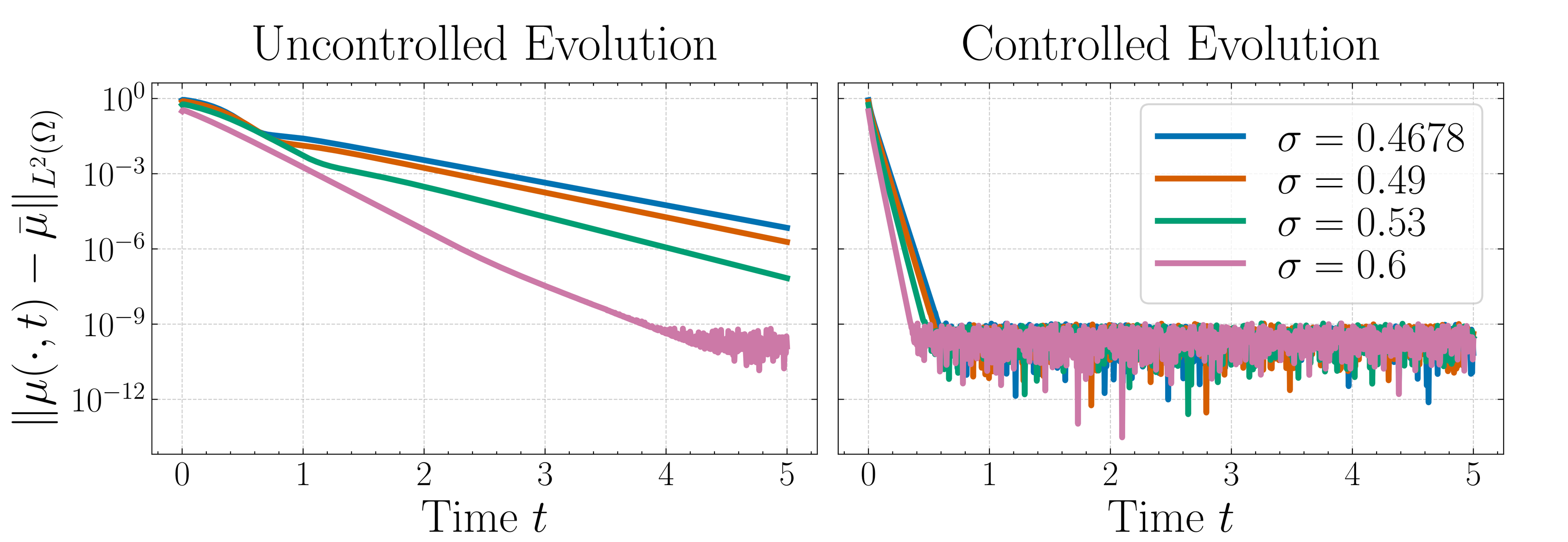

Figure 4: O(2) Kuramoto model with $V(x) = -\eta\cos(2\pi x)$, $W(x) = -\cos(2\pi x)$. The norm is taken in the weighted space $L^2(\Omega, \bar{\mu}^{-1}) = \{f : f\bar{\mu}^{-1/2} \in L^2(\Omega)\}$.

For small noise levels (σ), the uncontrolled system converges very slowly, while our feedback control enforces exponential decay at $\delta = 1$.

Conclusions and Take-Home

- ✓ **Interactions create structure:** slow mixing/convergence and metastability.
- ✓ **Stabilisation:** exponential convergence to the chosen equilibrium.
- ✓ Numerical method: spectral methods + Riccati feedback.
- ✓ **Next:** higher-dimensional solvers and robustness to perturbations.

Acknowledgements

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