IMPERIAL

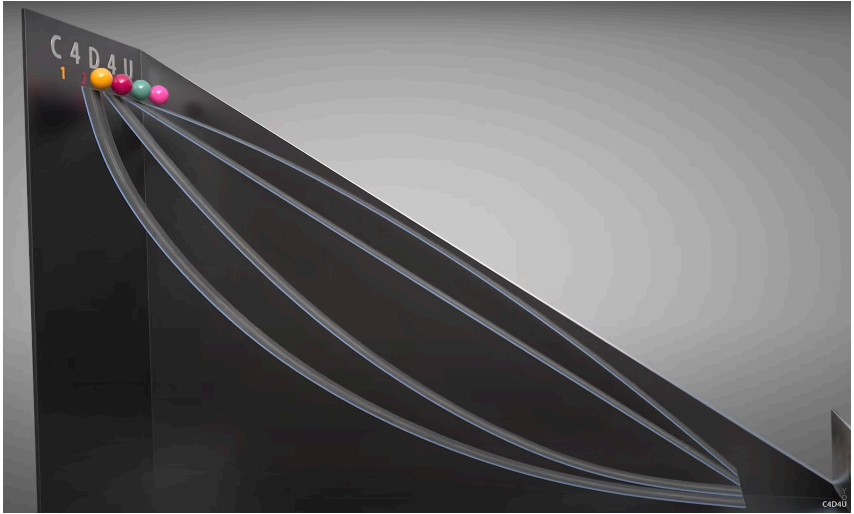
The Art of Controlling

How can we change how things behave?

Lucas M. Moschen

24/07/2025

How it all began...



What is this talk about?

A journey from classical mechanics to modern control theory. All with something in mind: modelling

We'll touch on:

- My **own** path into this field
- Real-world **motivations** from biology, engineering, and physics
- How mathematics helps us influence the **behaviour of systems**
- The core ideas behind **control theory**

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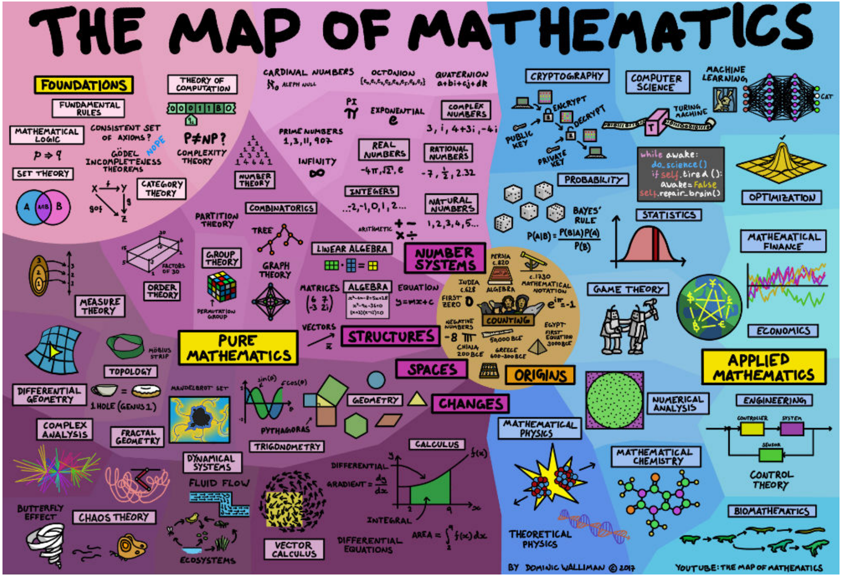
“The ultimate proof of our understanding of natural or technological systems is reflected in our ability to control them.”

Liu, Slotine & Barabási (2011)[1]

[1] Liu, Y.-Y., Slotine, J.-J., and Barabási, A.-L. Controllability of complex networks, Nature, 473, 167–173 (2011).


THE MAP OF MATHEMATICS
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FUNDAMENTAL RULES
MATHEMATICAL LOGIC
CONSISTENT SET OF AXIOMS?
GÖDEL INCOMPLETENESS THEOREMS
NOPE
SET THEORY
CATEGORY THEORY
THEORY OF COMPUTATION
P≠NP?
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INTEGERS
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CALCULUS
DIFFERENTIAL
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AREA = ∫f(x)dx
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ENCRYPT
DECRYPT
PUBLIC KEY
PRIVATE KEY
COMPUTER SCIENCE
MACHINE LEARNING
TURING MACHINE
PROBABILITY
BAYES' RULE
P(A|B) = P(B|A)P(A) / P(B)
while awake:
do_science()
if self.tired():
awake = False
self.repair_brain()
OPTIMIZATION
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MATHEMATICAL CHEMISTRY
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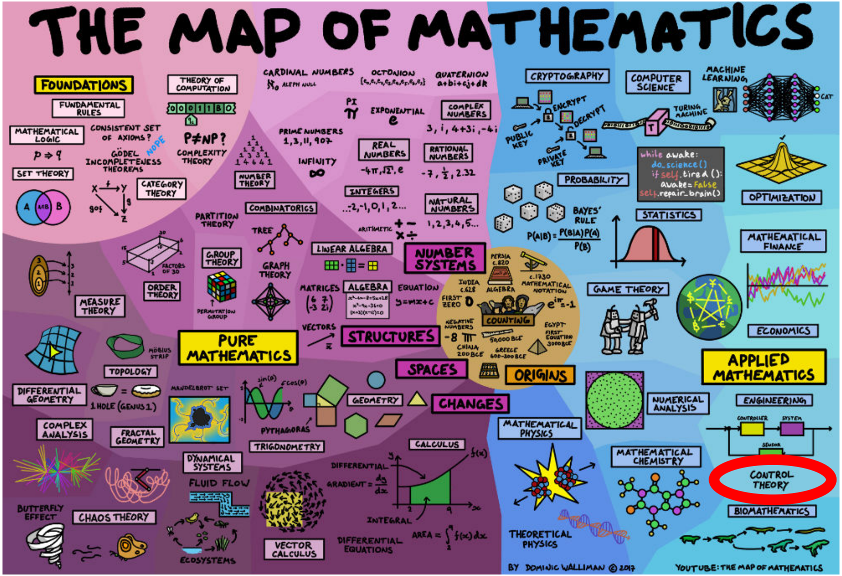
YOUTUBE: THE MAP OF MATHEMATICS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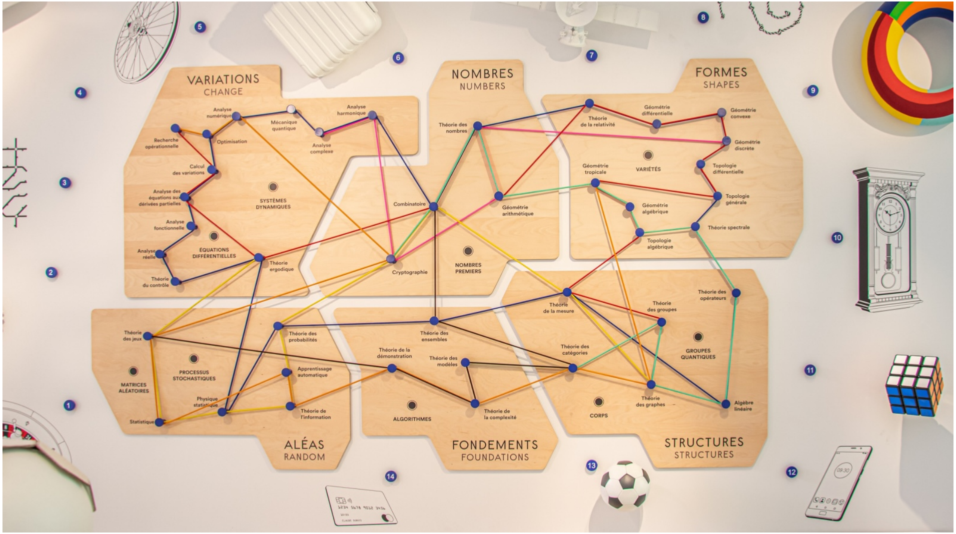
The mathematical landscape: a view from Paris



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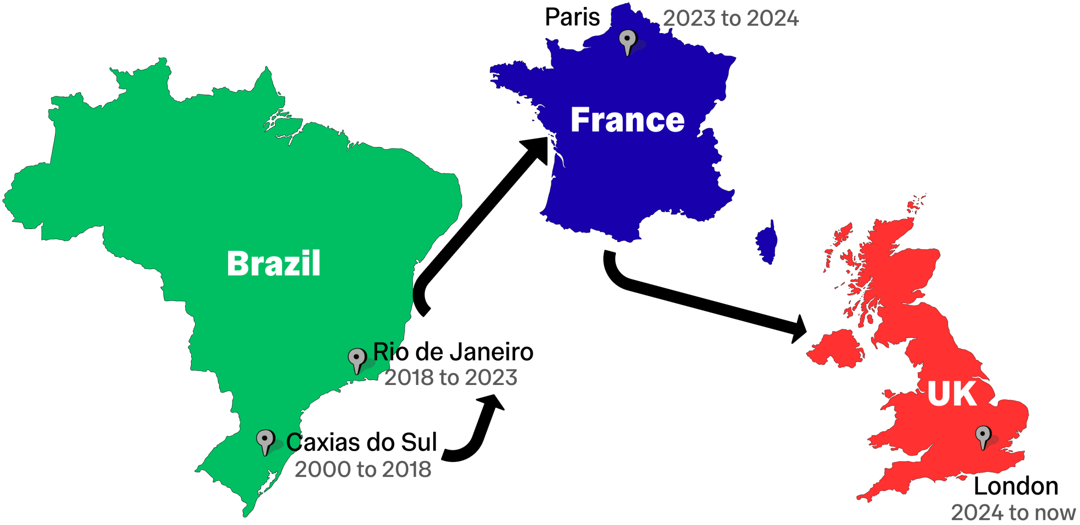
A bit about my path

Journey for the mathematics

A background on my studies



Journey for the mathematics

Challenges

- **Switching fields:** Adapting in different areas from mathematics
- **Cultural barriers:** Studying in Portuguese, French, and English
- **Imposter feelings:** Joining new research environments and learning to trust my ideas (also having them!)
- **Loneliness in research:** Building confidence when progress is slow and unclear
- **Communication:** Talking to peers in conferences and events

Journey for the mathematics

My academic mentors

Main Supervisors

M. Soledad Aronna (MSc)

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Co-Supervisors

Luis Almeida (MSc)

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M. Soledad Aronna

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- Particular application to COVID-19: quarantine, isolation, and testing
- Optimal distribution of vaccines in metropolitan areas
- **Areas:** Differential Equations, Optimisation, Functional Analysis, Statistics, Programming, ...

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And what is at the centre of all this?

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Control Theory.

Ceilidh Dance
From local rules to global behaviour

- A traditional Scottish dance with a caller who steers the choreography (pattern)
- Local interactions lead to global co-ordination
- We model the agents with stochastic differential equations (SDEs)
- The control is the caller, **changing the patterns**

Photo © Dave Conner (CC BY 4.0) – clip-art adaptation by L. Moschen.

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A glimpse on the history

The Art of Controlling
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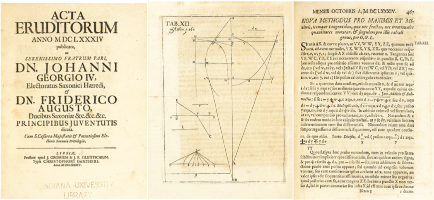
The Brachistochrone curve

Johann Bernoulli, 1696



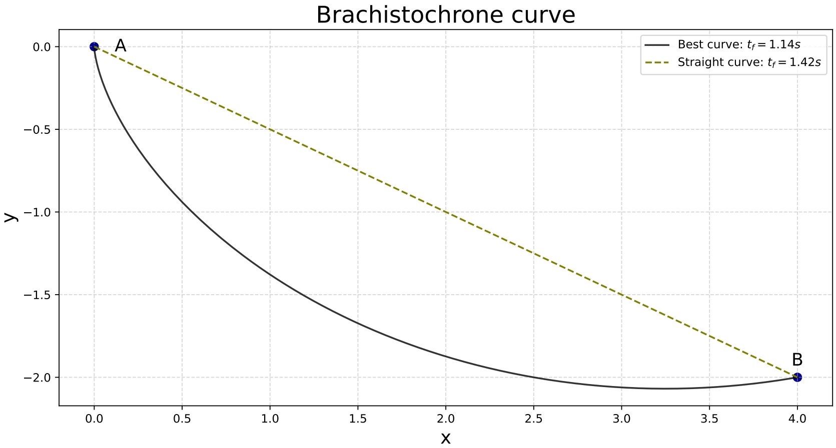
Johann Bernoulli (1667–1748)

"In a vertical plane, two points A and B are given. The task is to find the trajectory of a moving particle M such that, starting from point A and under the influence of its own weight, it reaches point B in the shortest possible time."[2]

[2] Johann Bernoulli, "Problema novum ad cujus solutionem Mathematici invitantur" (1696)

The Brachistochrone curve

Johann Bernoulli, 1696



Calculus of variations

The general problem in 1D

For each $x \in [a, b]$, we **choose** the value of the function $y'(x)$. The resulting curve should be **continuous**, start at y_0 and end at y_1, leading to

$$\min \int_a^b L(x, y(x), y'(x))dx$$

$$\text{s. t. } y(a) = y_0, \; y(b) = y_1.$$

From Calculus of Variations to Optimal Control

Calculus of Variations

$$\min / \max \int_a^b L(x, y(x), y'(x))\, dx$$

subject to y smooth

$$y(a) = y_0, \quad y(b) = y_1$$

$\Longrightarrow$

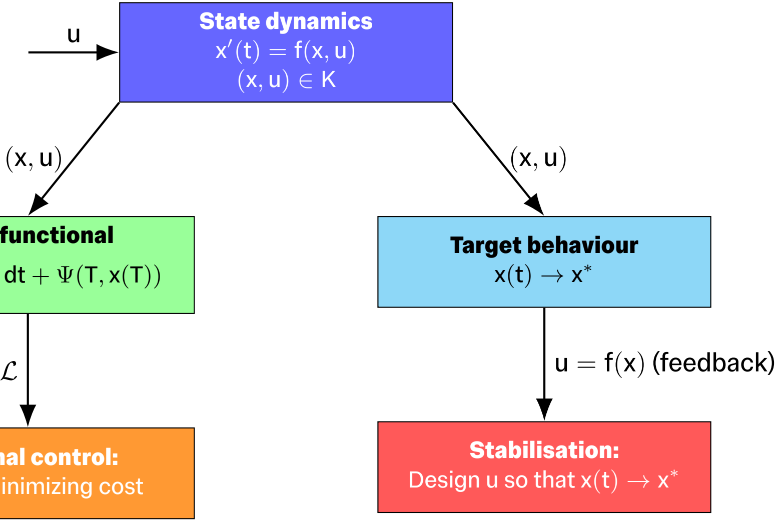
Optimal Control

$$\min / \max \int_0^T L(t, x(t), u(t))\, dt + \Psi(T, x(T))$$

subject to $\dot{x}(t) = f(t, x(t), u(t))$,

$$x(0) = x_0, \quad x(T) = x_1$$

Optimal control and stabilisation diagram



Control is everywhere
Applications across disciplines

Engineering: Controlling trajectories and combining sensor data in **self-driving cars**

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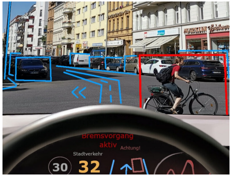

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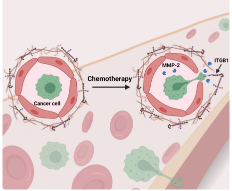

Biology: Personalised chemotherapy schedules that **maximise** effectiveness while **minimising** side effects

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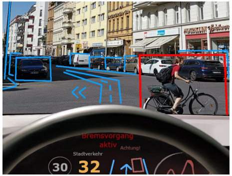
Applications across disciplines



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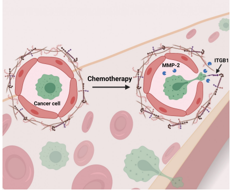

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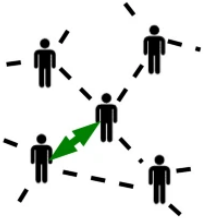

Social Sciences: Model and **influence** how opinions evolve in a society; for instance, in combating misinformation

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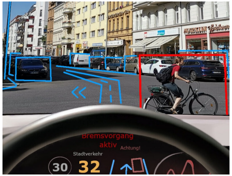
Applications across disciplines



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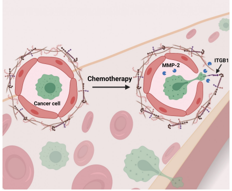

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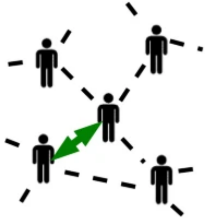

Social Sciences: Model and **influence** how opinions evolve in a society; for instance, in combating misinformation

Economics: Optimise **investment strategies** over time, balancing risk and return in dynamic markets

Let us practice a bit!

Fishing as a control problem

Problem

Given a fish population, we aim to maximize the fishing profit over a fixed time interval.

The first thing is: **model the problem**. What are the relevant variables?

$$x(t) := \text{number of fish in a lake at time } t$$

$$u(t) := \text{number of fish caught at time } t$$

Population dynamics (with fishing):

$$\dot{x}(t) = r\,x(t)\left(1 - \frac{x(t)}{k}\right) - u(t),$$

r: growth rate; $r\,x(t)/k$: death rate by competition

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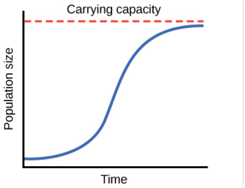

Logistic growth

What is the profit up to time T?

Each fish is sold at a price E, generating **revenue** $Eu(t)$. Fishing has an **associated cost**: the fewer fish in the lake, the harder it is to catch one. We model the unit cost as $c/x(t)$.

Therefore, the total profit over $[0, T]$ is

$$\text{Profit} := \int_0^T \left(Eu(t) - \frac{c}{x(t)} u(t) \right) dt$$

Complete optimal control problem:

$$\max \int_0^T \left(Eu(t) - \frac{c}{x(t)} u(t) \right) dt$$

subject to $\dot{x}(t) = rx(t) \left(1 - \frac{x(t)}{k} \right) - u(t)$,

$$0 \leq u(t) \leq U_{\max}, \quad x(t) \geq 0, \quad \text{for } t \in [0, T],$$

$$x(0) = x_0, \quad x(T) \geq x_{\min}.$$

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Interesting problems

Optimal Vaccination Strategies

- Epidemics are a complicated problem, especially in **metropolises**
- **Mathematical models** help describe and predict disease spread
- Vaccination is a reliable way to **control outbreaks** with minimal disruption
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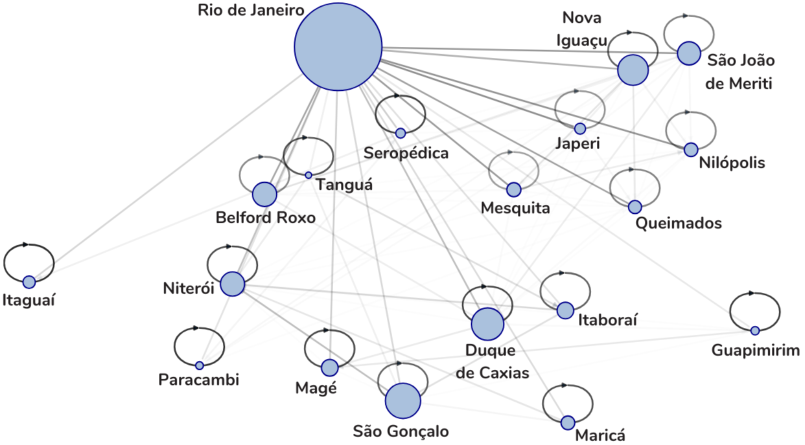
Epidemics in complex cities

Welcome to Rio de Janeiro metropolitan area!

- More people, more **contacts**.
- Intercity commuting increases the **spread**
- Vaccination should consider the **spatial distribution**



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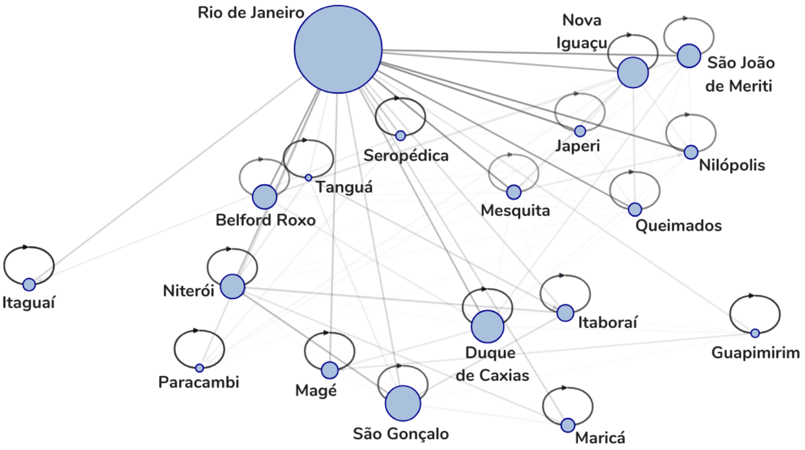

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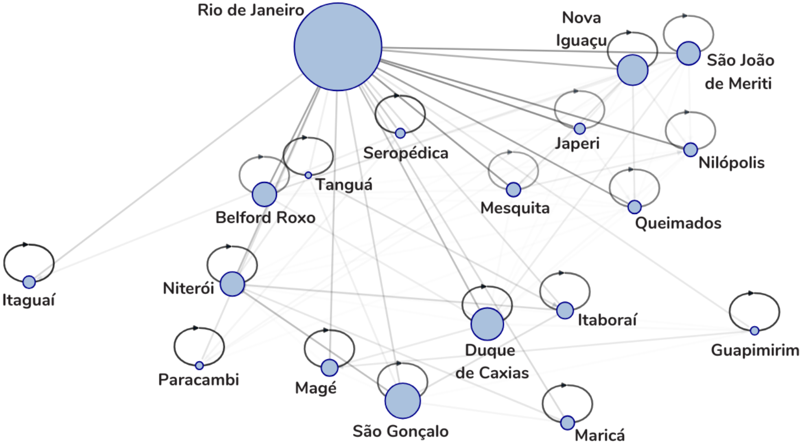
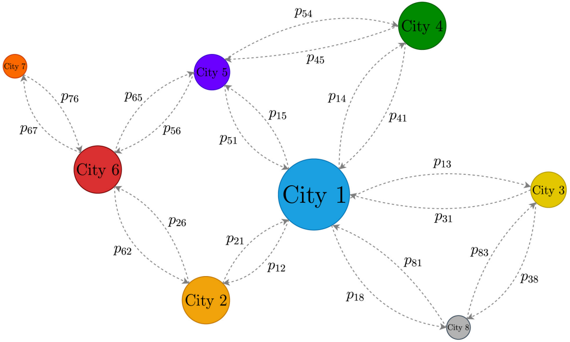
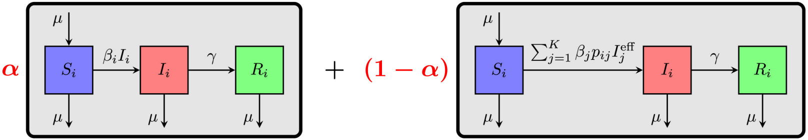


Modelling is the first task!

How commuting and epidemics interact?

- Cities are **nodes**, commuting are **connections**
- During the day, disease spreads **among** cities
- During the night, disease spreads **within** cities





What is the optimal vaccination strategy?

Aim

Use vaccination to reduce the burden of disease in a metropolitan area.

We add a **vaccination rate** $u_i(t)$ for each city, so

$$S_i'(t) = -\alpha\beta_i S_i(t) I_i(t) - (1-\alpha)\sum_{j=1}^{K} \beta_j (p_{ij} S_i(t)) I_j^{eff}(t) - u_i(t) S_i(t)$$

Trade-off: health impact (e.g., number of infected, number of deaths) $\times$ cost of vaccination.

$$\text{Burden of disease} := c_h \int_0^T \sum_{i=1}^{K} n_i I_i(t)\, dt + \text{Vaccination cost} := c_v \int_0^T \sum_{i=1}^{K} n_i u_i(t)\, dt$$

Vaccination constraints: $0 \leq u_i(t) S_i(t) \leq v_i^{max}$, $\int_0^t \sum_{i=1}^{K} n_i u_i(\tau) S_i(\tau)\, d\tau \leq V_{weekly}(t)$

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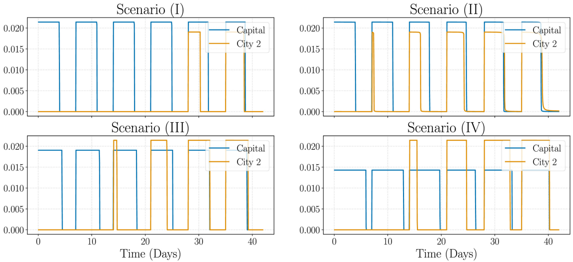
This problem has no closed-form solution. We solve it numerically.

From numerical experiments to theoretical results

Observation: simulations showed that the optimal vaccination strategy switches between **maximum effort** and **no effort**. This is known in control theory as a **bang-bang** strategy.



Theorem: For the optimal control problem with constraints, the function $u_i(t)S_i(t)$ is 0 or its maximum $v_i^{\max}$. Moreover, in each week it cannot grow!

Personally rewarding: the moment when numerics and theory align.

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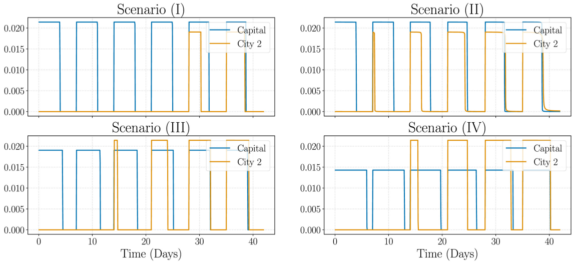

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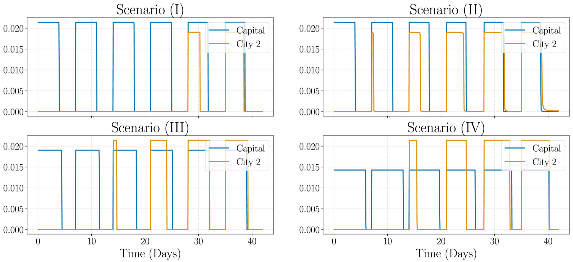

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What did we learn from vaccination control?

- Optimal control strategies reduce **infection spread**
- Modelling and optimisation make real-world policies more **effective**
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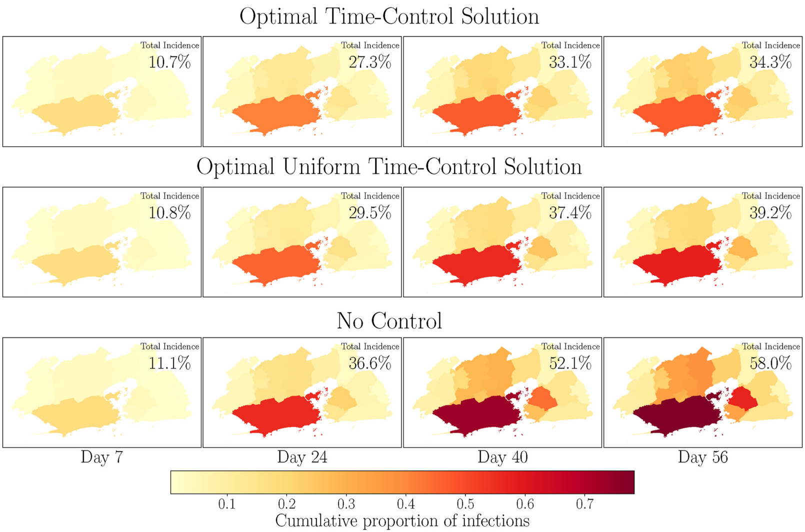

Controlling Mosquito Populations

- Mosquitoes are vectors of serious diseases like dengue, Zika, and chikungunya
- Population dynamics can be modelled mathematically to evaluate control strategies
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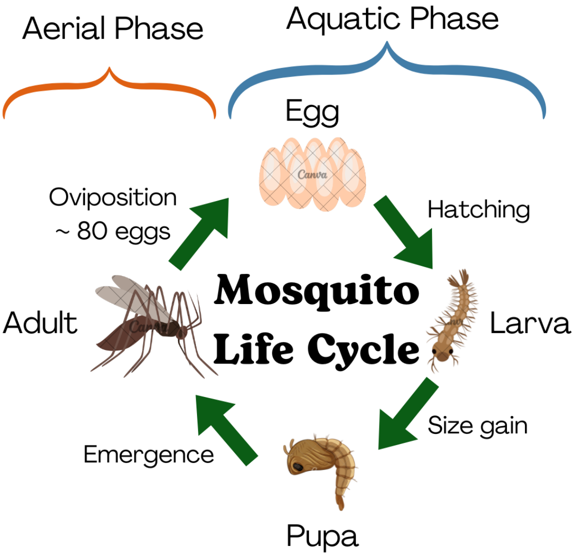
Mosquito life cycle

First we model!

- Each life stage will be a **compartment**, where we will **count** the number of individuals
- Parameters: birth, death, maturation, ...
- Described by a system of ordinary differential equations or stochastic processes



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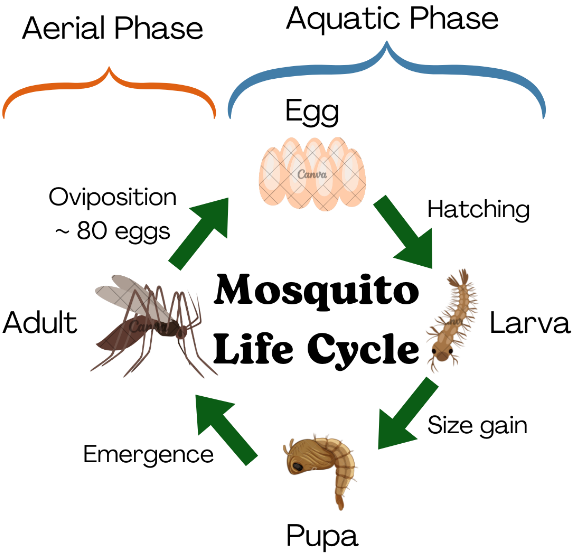

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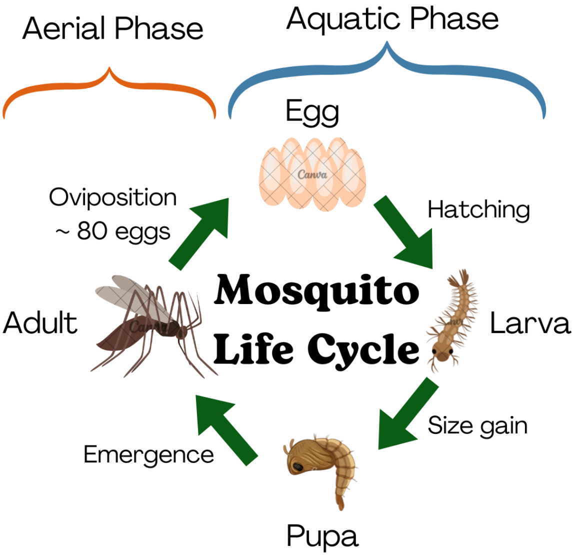

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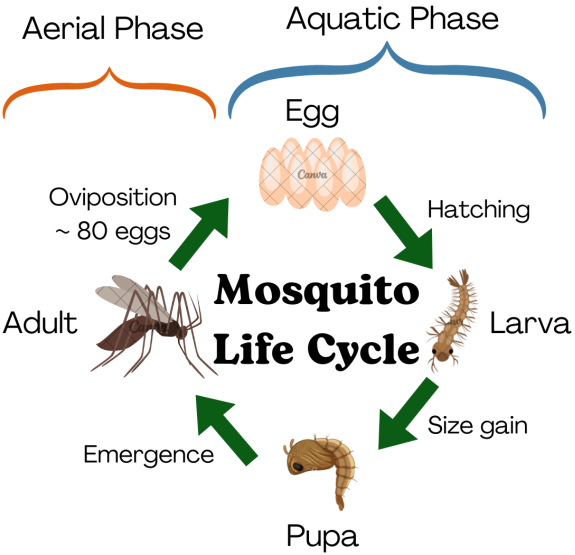
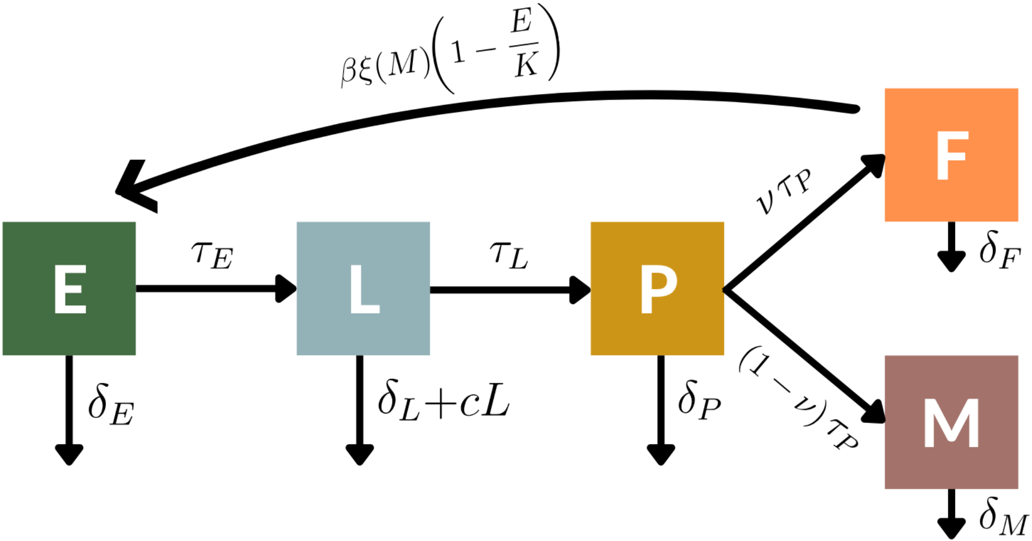
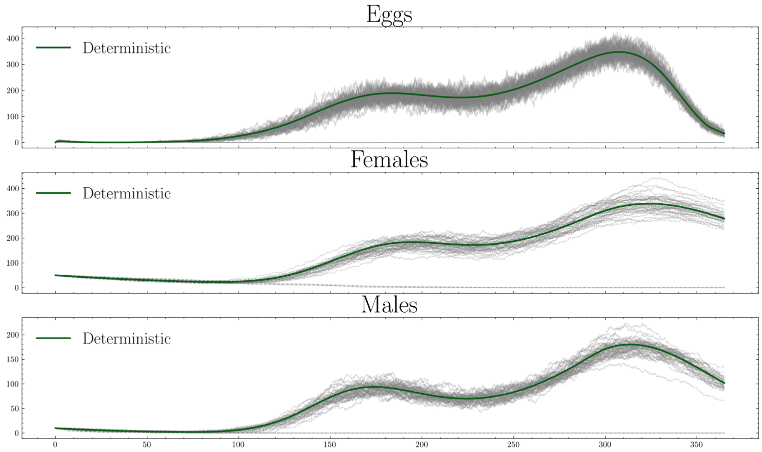


Modelling mosquito populations



Including uncertainty in mosquito dynamics



- **Stochastic version** of the model we saw
- Stochastic simulations capture **variation** and rare events
- **Temperature** also has a role here!
- **Control:** decrease the population level.

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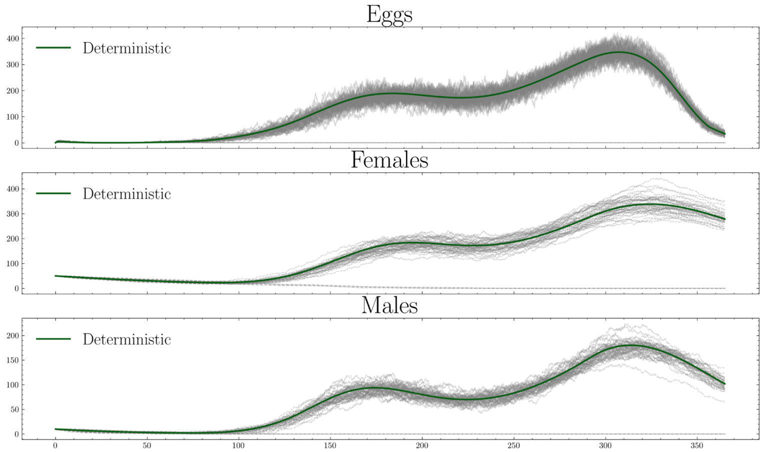

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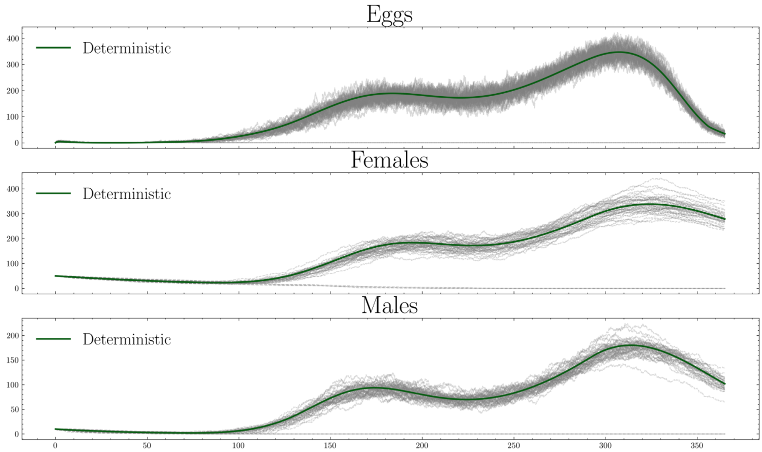

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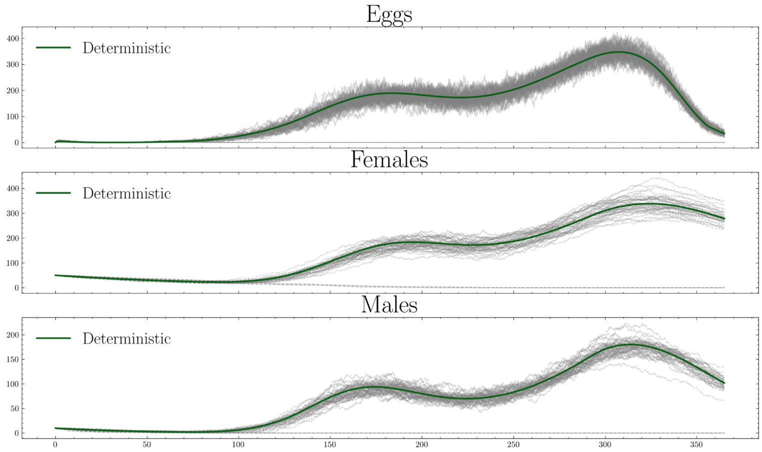

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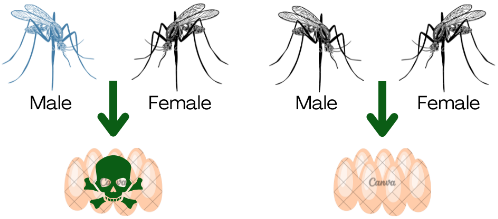
- Release of **sterile males** reduces future population
- Modelled as a control input in the system
- **Goal:** drive population to zero



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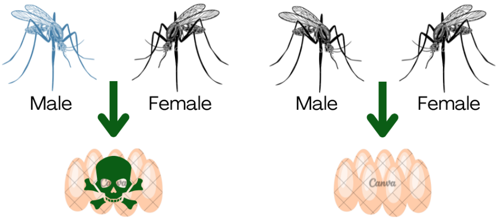

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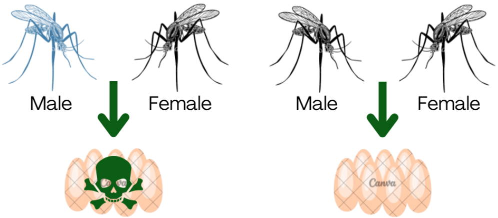

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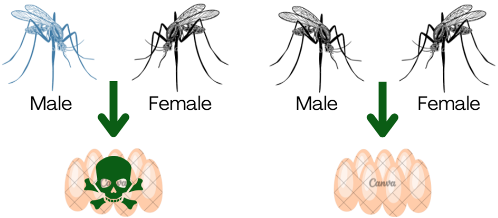
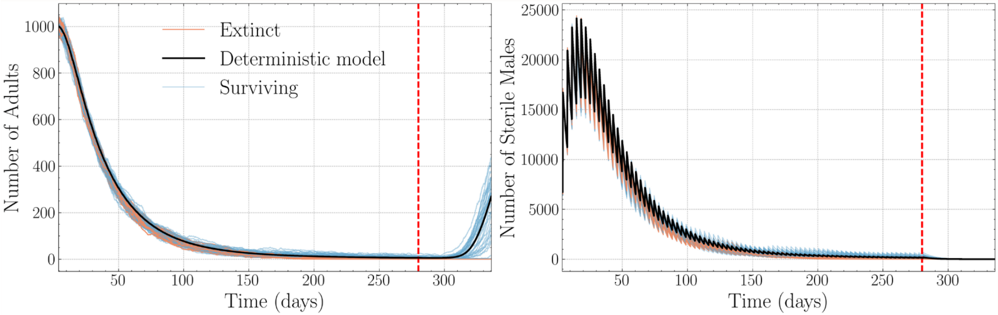
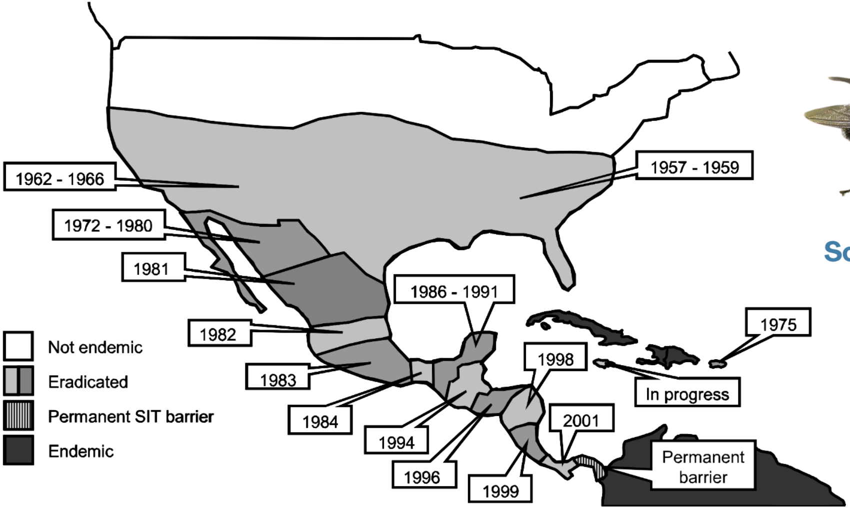




Success story: mosquito control in the Americas



Screwworm

Steering Particle Systems

- Systems with many interacting particles arise in physics, chemistry, and machine learning
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How to steer particles?

The long-term behaviour of particle systems can be slow.

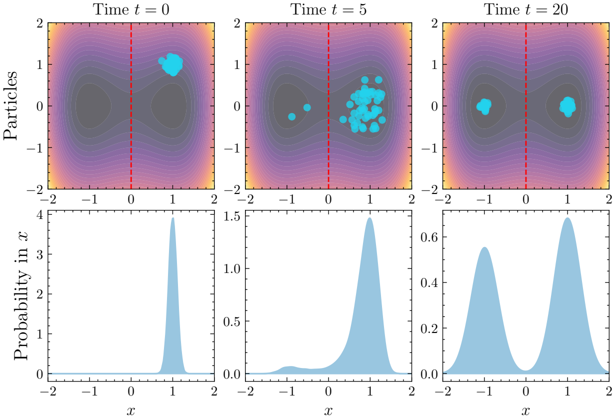

Figure: Particles in the energy landscape $V(x, y) = (x^2 - 1)^2 + y^2$

- **Stochastic particle systems**: models in molecular dynamics, Bayesian sampling, and collective behaviour.
- Often converges **slowly**, limiting efficiency.
- **Our aim:** develop control strategies to steer distributions towards desired targets, mainly changing the **long-term** behaviour.

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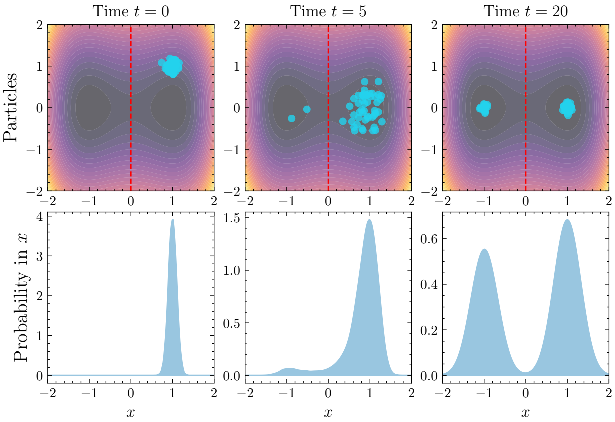

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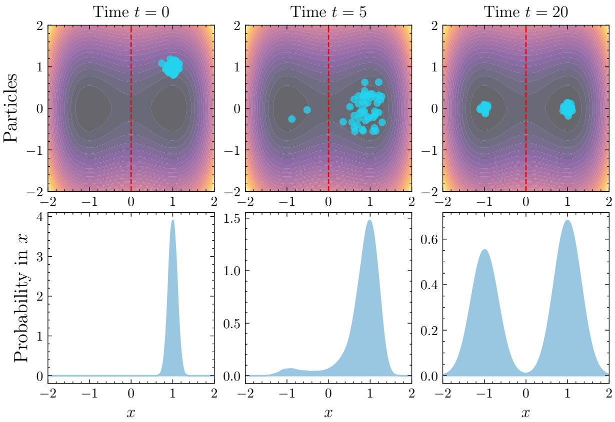

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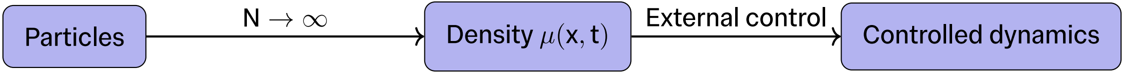
From particles to the mean-field limit and control



- Start with N interacting particles moving under random noise and forces.

- As N grows, approximate the particles by a density $\mu(t, x)$.
- μ evolves as a **known** equation

$$\partial_t \mu = \underbrace{\sigma \Delta \mu}_{\text{Diffusion}} + \underbrace{\nabla \cdot (\mu \nabla V)}_{\text{Energy drift}} + \underbrace{\nabla \cdot (\mu \nabla W * \mu)}_{\text{Interaction drift}}.$$

- Now we will control μ through control functions $u(t)$.

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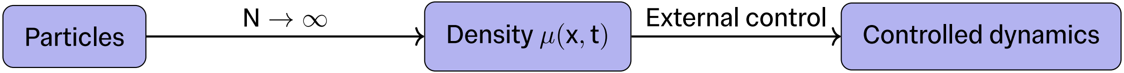

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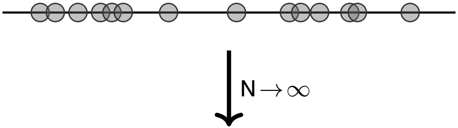

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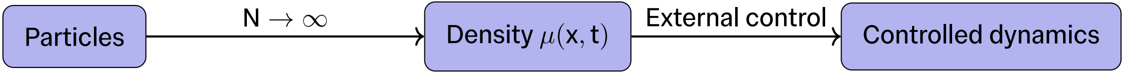

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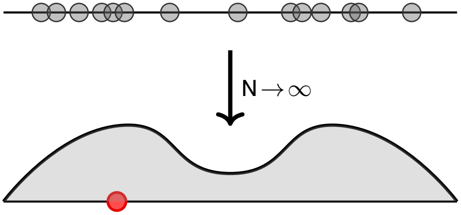

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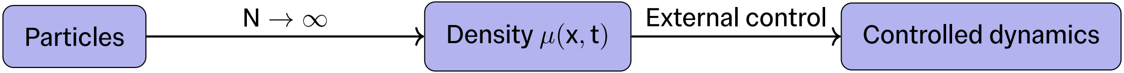

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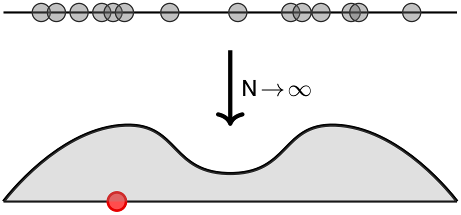

Feedback control accelerates convergence

How we use control to steer the dynamics as we wish

- Add an external force

$$V(x) \mapsto V(x) + \sum_{j=1}^{m} u_j(t)\, \alpha_j(x),$$

- The functions α_j indicate where to act and are chosen **smartly**. Time-signals u_j are chosen **optimally**.

- **Result:** controlled dynamics reach equilibrium far faster (or steer to new long-term states).

Feedback control accelerates convergence

How we use control to steer the dynamics as we wish

- Add an external force

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- **Result:** controlled dynamics reach equilibrium far faster (or steer to new long-term states).

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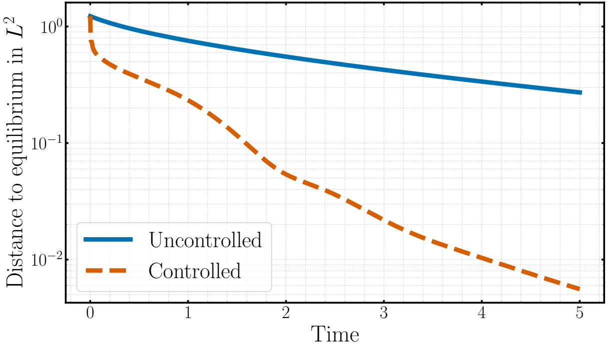

Figure: Convergence improvement for the double-well example.

Controlled noisy Kuramoto model
Removing the synchrony

- Coupled phase oscillators with sinusoidal interaction. Can exhibit synchronisation.
- For small K, noise dominates and the system remains desynchronised.
- For large K, oscillators synchronise; control can change this behaviour. The same idea can be applied to other systems, such as in **ecology**.

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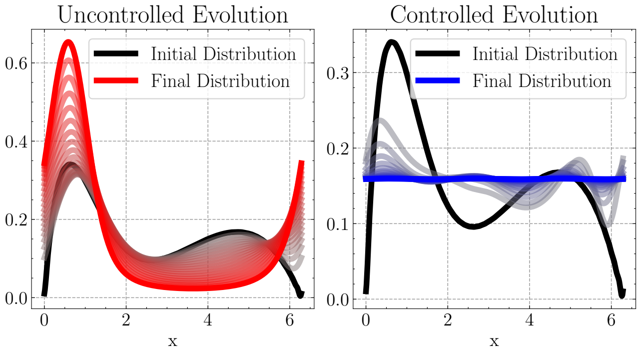

Figure: Consider $\Omega = [0, 2\pi)$, $V(x) = 0$, $W(x) = -K\cos(x)$

Take-aways and next steps

Key Take-Aways

- **PDE-based control** can accelerate convergence and reshape distributions.
- Simulations show the method **stabilises** different long-term states.

Next Steps

- Extend to high-dimensional and kinetic PDEs for real-world applications.
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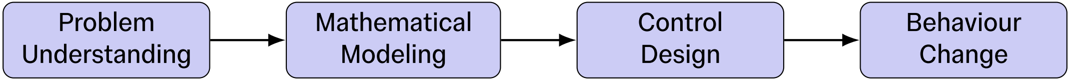
Wrapping up: the art of controlling



By **understanding** a system, **modelling** its dynamics, and designing the right **controls**, we can change how it **behaves**.

In every domain: epidemics, ecology, engineering, ...

IMPERIAL

Thank you! Questions?

A Brief Bibliography

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Let us see the maths behind the Brachistochrone curve.

- We look for a curve: $x \mapsto (x, y(x))$
- By the **Law of Conservation of Energy**

$$\frac{mv(x)^2}{2} - mgy(x) = 0 \implies v(x) = \sqrt{2gy(x)},$$

where v is the velocity of the ball, m is the mass, and g is the gravity constant

- The needed time is given by

$$T = \int_{s_a}^{s_b} \frac{ds}{v} = \int_a^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

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