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Optimal Control of Vaccination in Metapopulation Epidemics

Bang-Bang Optimal Control of Vaccination in Metapopulation Epidemics with Linear Cost Structures

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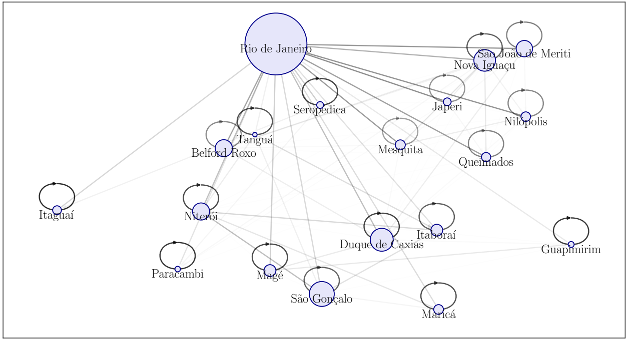
Vaccination on a Metropolitan Network

Rio de Janeiro metropolitan area: where CDC 2025 is being held

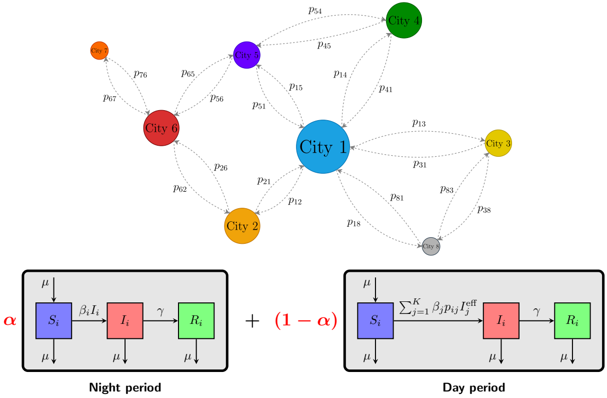


Commuting network for the Rio de Janeiro metropolitan area (node size = population, edge thickness = commuting flow).

- This is the Rio metropolitan area, where edges are daily commuting flows.
- During a pandemic, we must decide **where** and **when** to vaccinate under:
  - limited weekly shipments,
  - local application capacity.
- Previous works often use quadratic-cost optimal vaccination.
- **Questions:**
  - How does a **linear** vaccination cost change the optimal policy?
  - Can we characterise the structure of the optimal strategies under constraints?

Metropolitan Network Model[1]

Linear-cost vaccination on a commuting SIR network



Cities as nodes, directed edges = commuting flows.

- Each node is a city; during the day individuals **commute**, at night they **return home**.
- Each city has an SIR model; infections are driven by local and commuting contacts.
- Vaccination acts on the susceptible class S_i in each city, subject to shipment and capacity constraints. So

$$S_i'(t) = (\text{infection terms}) - u_i(t)\, S_i(t).$$

[1] M. Aronna and L. Moschen (2024). "Optimal vaccination strategies on networks and in metropolitan areas."

Vaccination Control and Observed Structure

From metropolitan numerics to a structural question

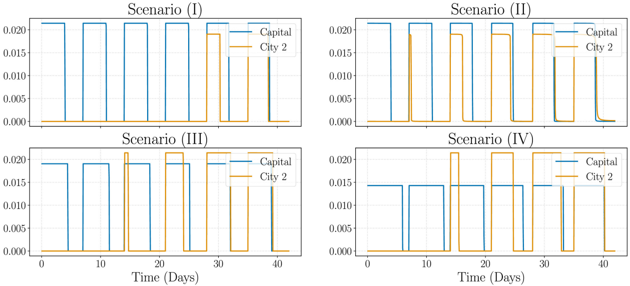
- Linear cost functional:

$$J(u) = c_h \int_0^T \sum_{i=1}^{K} n_i I_i(t)\, dt + c_v \int_0^T \sum_{i=1}^{K} n_i u_i(t)\, dt.$$

- Constraints:

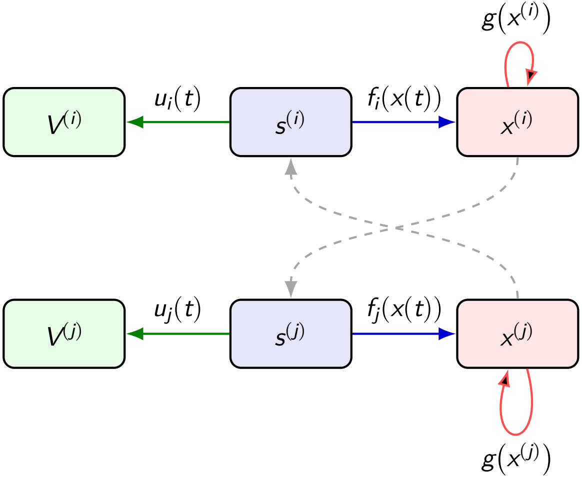
$$0 \le u_i(t) S_i(t) \le v_i^{\max}, \qquad \sum_{i=1}^{K} V_i(t) \le V_{\text{weekly}}(t).$$

- **Numerical observation**: optimal controls switch between **maximum effort** and **no effort** in each week
- Can we prove and generalise this **bang-bang** structure for a broader class.



General Metapopulation Epidemic Model

Compartments, controls, and interactions



- K **groups**: cities, regions, or classes. For each group i, we track $s^{(i)}$, $x^{(i)} \in \mathbb{R}^d$, and $V^{(i)}$.
- $x^{(i)}$ collects all **non-susceptible stages** of the disease in group i.
- Individuals in j can **infect** those in i, and vice versa. **Disease progression** inside i is described by $g\big(x^{(i)}(t)\big)$.
- The vaccination rate $u_i(t)$ moves individuals from $s^{(i)}$ to $V^{(i)}$ subject to constraints.

Quadratic vs Linear Cost Structures
From smooth controls to bang–bang policies

Quadratic vaccination cost

$$J_{\mathrm{q}}[u] = \text{disease cost}(x) + c_v \int_0^T \sum_{i=1}^K n_i\, u_i(t)^2\, dt.$$

- L^2-regularisation of u:
  - large u_i heavily penalised,
  - favours **smooth, moderate** controls.
- Nice analysis (strict convexity, unique minimiser), but **marginal cost per dose increases with rate**.
- Often used for convenience rather than realism.

Linear vaccination cost (this work)[2][3]

$$J[u] = \text{disease cost}(x) + c_v \int_0^T \sum_{i=1}^K n_i\, u_i(t)\, s^{(i)}(t)\, dt.$$

- **Constant marginal cost per dose**: each vaccinated individual costs c_v.
- Combined with capacity & shipment constraints: natural **on/off** decisions.
- Our result: optimal policies are **bang-bang** (at most one switch per week) $\Rightarrow$ simpler interpretation.

[3] H. Behncke (2000). “Optimal control of deterministic epidemics.”

[3] M. de Pinho, I. Kornienko, and H. Maurer (2015). “Optimal control of a SEIR model with mixed constraints and L1 cost.”

Main Results I

Well-posedness and optimality conditions

- **Problem:** for a metapopulation system with vaccination controls $u_i(t)$, we minimise $J[u]$ subject to the dynamics and
$$0 \le u_i(t)\, s^{(i)}(t) \le v_i^{\max}, \qquad \sum_{i=1}^{K} V^{(i)}(t) \le D_\varepsilon(t)$$
(mixed control-state constraint + pure-state constraint).

- **Well-posed dynamics** and **existence.** Under mild assumptions on f_i and g (Lipschitz, essential non-negativity): the system is well-posed (unique bounded solution, positivity preserved) and the problem admits a **global minimiser** (Cesari-type existence).

- **Pontryagin Maximum Principle with constraints.**[4][5]
  - Adjoint variables $\psi_s^{(i)}, \psi_x^{(i)}, \psi_V$ and multipliers for the capacity and shipment constraints.
  - The maximum principle leads to **switching functions** $\varphi_i(t)$ that determine whether u_i is 0 or maximum.
  - The next step is to show they generate **bang-bang policies** with at most one switch per week.

[4] M. Biswas, L. Paiva, M. de Pinho, and et al. (2014). "A SEIR model for control of infectious diseases with constraints."

[5] A. Arutyunov, D. Karamzin, and F. Pereira (2011). "The maximum principle for optimal control problems with state constraints."

Structural Assumptions
Why we restrict to linear / affine models

- **Model class.** Metapopulation SIR/SEIR-type models[6,7] where

$$f_i(x),\ g(x^{(i)}),\ f_0(x) \text{ are linear or affine in the state,}$$

  but we keep the non-linearity $s^i f_i(x(t))$. This covers the network models we have in mind.

- **In the proofs.** This structure makes the adjoint equations and

$$\dot{\varphi}_i(t) = \frac{d}{dt}\frac{\partial H}{\partial u_i}$$

  explicit and **sign-controlled**, which is used to show monotonicity of φ_i and rule out singular arcs.

[6] J. Lemaitre et al. (2022). "Optimal control of the spatial allocation of COVID-19 vaccines: Italy as a case study."
[7] L. Nonato et al. (2002). "Robot Dance: A mathematical optimization platform for intervention against COVID-19 in a complex network."

Main Results II

Bang–bang policies with one switch per week

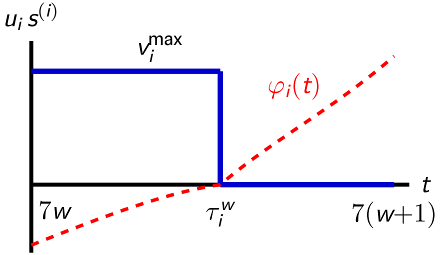
Structural theorem (informal)

For each group i and each week w, any optimal control is

$$u_i(t)s^{(i)}(t) = \begin{cases} v_i^{\max}, & 7w \le t < \tau_i^w, \\ 0, & \tau_i^w < t \le 7(w+1), \end{cases}$$

for some switching time τ_i^w. **Full speed then stop, at most once per week.**

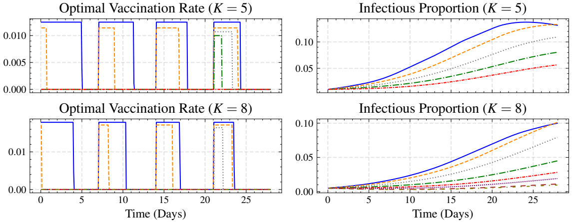
- Maximum principle: $\varphi_i(t) > 0 \Rightarrow u_i(t) = 0$, $\varphi_i(t) < 0 \Rightarrow u_i(t)s^{(i)}(t) = v_i^{\max}$.
- φ_i is **monotone within each week** $\Rightarrow$ at most one zero per week (no singular arcs).
- **Consequence:** the infinite-dimensional problem reduces to optimising over $\{\tau_i^w\}$.



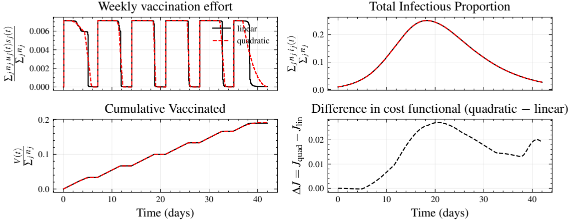
Numerical Illustration

Metapopulation SIR networks with weekly shipments

- **Set-up:** SIR metapopulation model with $K = 3, 5, 8$ cities.
- **Numerics:** first discretise, then optimise.
- **What we see in practice:**
  - confirms the theoretical results.
  - vaccinate first and more in the most contagious cities.
  - linear and quadratic costs are similar, but linear cost exposes the bang-bang structure.



Optimal weekly vaccination and infectious curves for $K = 3, 5, 8$.



Linear vs quadratic cost for $K = 5$.

Ideas Behind the Proofs

From Pontryagin's principle to bang-bang structure

- **Well-posedness and invariance**
  - Carathéodory theory $\Rightarrow$ unique **bounded solution**.
  - Metzler / **non-negative structure** $\Rightarrow$ state stays in the positive orthant.
  - Gronwall estimate $\Rightarrow$ $s^{(i)}(t)$ uniformly bounded away from 0.
- **Pontryagin with state and mixed constraints**
  - Hamiltonian with adjoints and multipliers (capacity, shipments).
  - Maximisation condition $\Rightarrow$ switching functions $\varphi_i(t)$.
- **Monotone switching functions**
  - Sign structure of the adjoint system $\Rightarrow$ $\dot{\varphi}_i(t)$ **has fixed sign** on each week.
  - **No singular arcs**: the set $\{\varphi_i = 0\}$ has measure zero.
- **Bang-bang with one switch per week**
  - Monotone φ_i $\Rightarrow$ at most **one zero-crossing** per week.
  - Hence $u_i(t)s^{(i)}(t) \in \{0,\, v_i^{\max}\}$ with a single switching time in each week.

Take-Home Messages

Structure and practicality of optimal vaccination

- **Linear cost + realistic constraints** $\Rightarrow$ **bang-bang** optimal vaccination in metapopulation models.
- **Simple structure:** each group vaccinates at full capacity, then stops once per week (single switching time).
- **Practical payoff:** functional optimisation reduces to a few switching times, enabling leaner algorithms and clearer policy design.
- **Beyond the linear/affine case:** numerically, the bang-bang behaviour appears for some non-linear $f_i(x(t))$, this suggests **more general results** are within reach, but the theory is still open.
- **Personally:** part of the appeal here starts from the numerical behaviour, uncovering the underlying structure, and then seeing how far that picture can be pushed beyond the initial assumptions.

IMPERIAL

Thank you for your attention!