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Latin American Congress on Industrial and Applied Mathematics

Feedback Control of Mean-Field PDEs: The McKean-Vlasov Equation

Dante Kalise[1], Lucas M. Moschen[1], and Greg Pavliotis[1]

[1] Imperial College London

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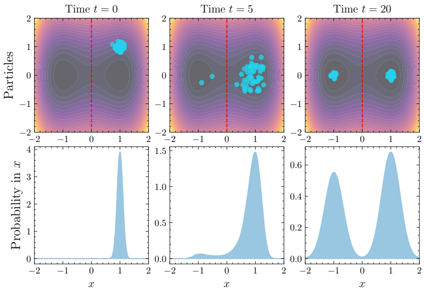
Controlling the Behaviour of Particle Systems

Photo © Dave Conner (CC BY 4.0) – clip-art adaptation by L. Moschen.

- Feedback metaphor: in a **Scottish ceilidh**, a caller guides the group's choreography
- Systems with many interacting particles arise in physics, chemistry, and machine learning
- Their collective dynamics can be described by SDEs and PDEs
- **Goal:** design feedback controls that steer the *distribution* toward a desired long-time behaviour.

Sampling Viewpoint

The long-term behaviour of particle systems can be slow[1]



Particles in the energy landscape $V(x, y) = (x^2 - 1)^2 + y^2$.

Target (Gibbs) law: $\pi(dx) \propto e^{-V(x)/\sigma}$

- **Overdamped Langevin dynamics:**

$$dX_t = -\nabla V(X_t)\, dt + \sqrt{2\sigma}\, dB_t, \quad \mu_t = \mathrm{Law}(X_t)$$

- If π satisfies a *log-Sobolev inequality*, then

$$\|\mu_t - \pi\|_{L^2(\pi^{-1})} \leq e^{-\lambda t} \|\mu_0 - \pi\|_{L^2(\pi^{-1})}.$$

For nonconvex V, λ can be small $\Rightarrow$ **slow convergence**[2]

- With interactions (collective effects), for $i = 1, \ldots, N$,

$$dX_t^{i,N} = -\nabla \left(V(X_t^{i,N}) + \frac{1}{N} \sum_{j=1}^{N} W(X_t^{i,N} - X_t^{j,N}) \right) dt + \sqrt{2\sigma}\, dB_t^i,$$

which can lead to **multiple macroscopic equilibria**

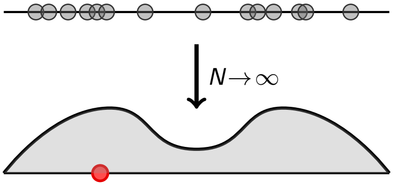
[1] Carrillo, Gvalani, Pavliotis, Schlichting (2020).

[2] Lelievre, Nier, Pavliotis (2013)

From Particles to the Mean-Field Limit and Control

Particles $\xrightarrow{N \to \infty}$ Density $\mu(x, t)$ $\xrightarrow{\text{External control}}$ Controlled dynamics

- N interacting particles moving under noise and forces
- Approximate the particles by a density $\mu(t, x)$ for large N
- μ evolves as a **limit PDE (mean-field limit)**



$$\partial_t \mu = \underbrace{\sigma \Delta \mu}_{\text{Diffusion}} + \underbrace{\nabla \cdot (\mu \nabla V)}_{\text{Energy drift}} + \underbrace{\nabla \cdot (\mu \nabla W * \mu)}_{\text{Interaction drift}}$$

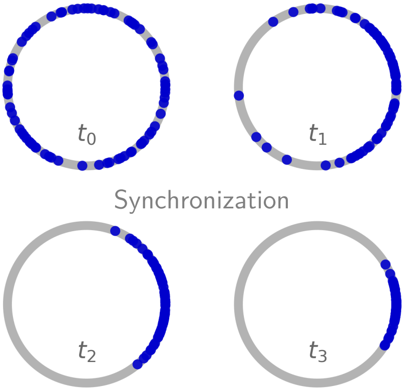
We design a feedback control at the PDE level to accelerate and stabilise mean-field particle systems[3]

[3] Fornasier, Solombrino (2014), Albi, Choi, Fornasier, Kalise (2018).

Project Map and Main Objective

Particles $\rightarrow$ Mean-field PDE $\rightarrow$ Spectral analysis $\rightarrow$ Feedback control Stabilisation $\rightarrow$ Finite-N control

- **Steady states:** $\bar{\mu} \propto \exp\left\{-\frac{1}{\sigma}(V + W * \bar{\mu})\right\}$
- **Convex potentials:** convexity of V and W implies uniqueness of $\bar{\mu}$ and $\mu(t,\cdot) \xrightarrow{L_1} \bar{\mu}$ exponentially
- **Nonconvex potentials:** multiple steady states may emerge[4].



[4] Bicego, Kalise, Pavliotis (2025).

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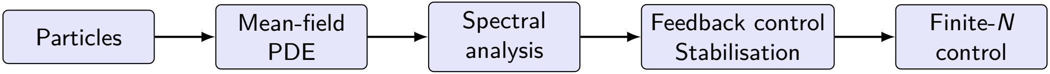

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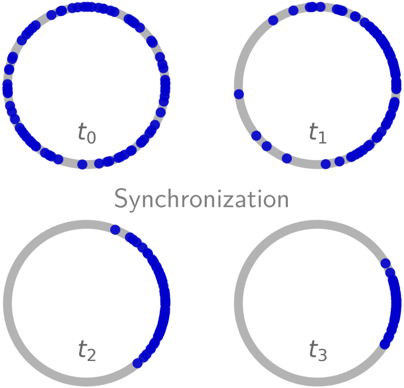


Objective

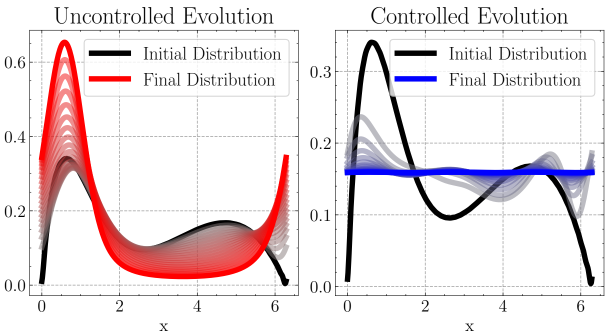
- **Choose $\bar{\mu}$:** local stabilisation around $\bar{\mu}$ with rate δ
- Low-dimensional feedback control based on μ_t:

$$\|\mu_t - \bar{\mu}\|_{L^2(\bar{\mu}^{-1})} \leq Ce^{-\delta t}\|\mu_0 - \bar{\mu}\|_{L^2(\bar{\mu}^{-1})}$$

[4] Bicego, Kalise, Pavliotis (2025).

Noisy Kuramoto Model
Stabilising an unstable equilibrium

- Mean-field model on $\mathbb{T}$ with sine coupling[4]
- The uniform steady state can become **unstable** as coupling increases (multiple equilibria)[5]
- **Feedback** stabilises the desired steady state
- For this model, $\bar{\mu} = 1/(2\pi)$ is stable for $K \le 1$ and loses stability for $K > 1$



Example: $W(x) = -K\cos x$, $V = 0$

[5] Bertini, Giacomin, Pakdaman (2010).

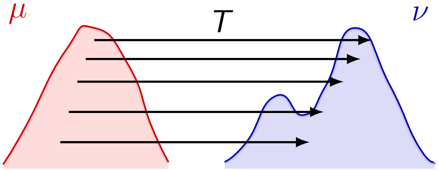
Connecting with Probability Spaces

Distance between distributions via moving mass

- We work on $\mathcal{P}_2(\Omega)$, probability measures with finite second moment
- The quadratic Wasserstein distance is the **optimal transport cost**[6]

$$\mathrm{W}_2^2(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \iint_{\Omega\times\Omega} |x-y|^2 \, d\gamma(x,y),$$

where $\Pi(\mu,\nu)$ are couplings with marginals μ and ν



arrows represent $(x, T(x))$

$\gamma = (\mathrm{Id}, T)_{\#}\mu \in \Pi(\mu,\nu)$

- Introduce a functional $\mathcal{F} : \mathcal{P}_2(\Omega) \to \mathbb{R}$. The mean-field PDE is the W_2-gradient flow of $\mathcal{F}$:

$$\partial_t \mu_t = \nabla \cdot \Big(\mu_t \nabla \overbrace{\frac{\delta \mathcal{F}}{\delta \mu}(\mu_t)}^{\text{First variation}} \Big), \qquad \frac{d}{dt}\mathcal{F}(\mu_t) = -\int_\Omega \left| \nabla \frac{\delta \mathcal{F}}{\delta \mu}(\mu_t) \right|^2 d\mu_t \le 0$$

[6] Jordan, Kinderlehrer, Otto (1998), Benamou, Brenier (2000), Villani (2009).

Wasserstein Gradient Flows for Mean-Field Models

Free energy $\mathcal{F} = \mathcal{E} + \sigma$ Ent and descent dynamics

- Energy $+$ entropy (on $\Omega = \mathbb{T}^d$, similarly for bounded $C^{1,1}$ domains):

$$\mathcal{F}(\mu) = \underbrace{\int_\Omega V(x)\, d\mu(x) + \frac{1}{2} \iint_{\Omega\times\Omega} W(x-y)\, d\mu(x)\, d\mu(y)}_{\mathcal{E}(\mu)} + \sigma \underbrace{\int_\Omega \mu(x) \log \mu(x)\, dx}_{\mathrm{Ent}(\mu)}$$

- McKean-Vlasov / Fokker-Planck as a W_2-gradient flow[7]:

$$\partial_t \mu_t = \nabla \cdot \left(\mu_t \nabla \frac{\delta \mathcal{F}}{\delta \mu}(\mu_t) \right) = \sigma \Delta \mu_t + \nabla \cdot (\mu_t \nabla V) + \nabla \cdot (\mu_t \nabla (W * \mu_t))$$

- Stationary states satisfy $\frac{\delta \mathcal{F}}{\delta \mu}(\bar{\mu}) = \text{const}$. **Linearising the gradient flow at** $\bar{\mu}$ yields a self-adjoint operator A in $H^1(\bar{\mu}) \cap \{\varphi : \int_\Omega \varphi d\bar{\mu} = 0\}$ with **compact resolvent** (hence discrete spectrum):

$$\partial_t \varphi = -A\varphi, \qquad \langle \varphi, \psi \rangle_{\bar{\mu}} := \int_\Omega \nabla\varphi \cdot \nabla\psi\, d\bar{\mu}, \qquad \mu - \bar{\mu} = \mathcal{I}_{\bar{\mu}}\varphi := -\nabla \cdot (\bar{\mu} \nabla \varphi)$$

[7] Carrillo, McCann, Villani (2003).

Control as Shaping the Free Energy

- **Goal:** stabilise an equilibrium $\bar{\mu}$ of the McKean-Vlasov equation via **feedback control**
- **How:** Modify V as[8]

$$V(x) \mapsto V(x) + \overbrace{\sum_{j=1}^{m} u_j(t)\,\alpha_j(x)}^{\text{Control input}} \iff \mathcal{F}_u(\mu,t) = \mathcal{F}(\mu) + \sum_{j=1}^{m} u_j(t) \int_\Omega \alpha_j \, d\mu$$

- **Closed-loop mean-field dynamics:**

$$\partial_t \mu_t = \nabla \cdot \left(\mu_t \nabla \Big(\frac{\delta \mathcal{F}}{\delta \mu}(\mu_t) + \sum_{j=1}^{m} u_j(t)\alpha_j \Big) \right) \Rightarrow \text{Bilinear control} + \text{Nonlocal/nonlinear}$$

- **Strategy:** fix $\delta > 0$.

Spectral analysis of linearised operator $-A$ → Identify unstable modes ($\lambda \geq -\delta$) → Use LQR to build a finite-dim feedback → Closed-loop stability Contraction argument

[8] Breiten, Kunisch, Pfeiffer (2018).

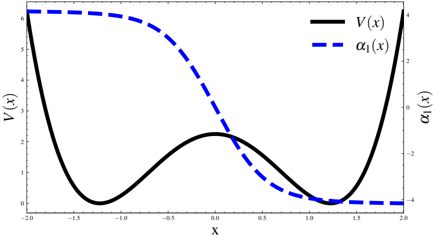
Feedback Design: Spectral Mode Targeting + Riccati Feedback

- Set $y = \mathcal{I}_{\bar{\mu}}\varphi := -\nabla\cdot(\bar{\mu}\nabla\varphi)$. Then, the (uncontrolled) linearized dynamics is $\partial_t y = \mathcal{L}y$ with $\mathcal{L} = -\mathcal{I}_{\bar{\mu}} A \mathcal{I}_{\bar{\mu}}^{-1}$
- The controlled linearised dynamics takes the form

$$\partial_t y = \mathcal{L}y + \mathcal{B}u, \quad \mathcal{B}u = \sum_{j=1}^{m} \mathcal{B}_j u_j, \quad \mathcal{B}_j = \nabla\cdot(\bar{\mu}\nabla\alpha_j).$$

- Fix $\delta > 0$. Only **finitely many eigenvalues** of $\mathcal{L}$ satisfy $\Re\lambda \geq -\delta$, hence only finitely many modes can prevent δ-decay
- Mode targeting: let $\psi_1, \ldots, \psi_m$ be eigenfunctions of $\mathcal{L}$ with $\Re\lambda_j \geq -\delta$; choose α_j via

$$\mathcal{B}_j = \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \psi_j, \qquad j = 1, \ldots, m$$



Under this setting, the pair $(\mathcal{L}, \mathcal{B})$ satisfies a δ-Hautus condition, hence is δ-stabilisable

Riccati Feedback for the Linearised System
Infinite-dimensional LQR on the zero-mean subspace

- **LQR cost with exponential weight:**

$$J(u) = \frac{1}{2}\int_0^\infty e^{2\delta t}\big(\langle y, \mathcal{M}y\rangle + \|u\|^2\big)\, dt, \quad \mathcal{M} \succ 0$$

- **Optimal feedback:** $u(t) = -\mathcal{B}^*\Pi\, y(t)$, where Π solves the algebraic Riccati equation[9]

$$(\mathcal{L}^* + \delta I)\Pi + \Pi(\mathcal{L} + \delta I) - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{M} = 0$$

- **Stabilisation guarantee:** if the δ-Hautus condition holds, then $\|y(t)\| \le Ce^{-\delta t}\|y_0\|$

[9] Curtain, Zwart (2005).

From General Gradient Flows to McKean–Vlasov

Spectral analysis in the specific Hilbert setting

- **Focus**[10]**:** the McKean–Vlasov equation on $\mathbb{T}^d$.
- **Transformed operator:**

$$\mathcal{H} := -\mathcal{U}\mathcal{L}\mathcal{U}^{-1} = -\sigma\Delta + \Psi(x) + \mathcal{K}$$

  where Ψ is an effective potential, $\mathcal{K}$ is a compact integral operator and

$$\mathcal{U} : L^2(\Omega, \bar{\mu}^{-1}) \to L^2(\Omega), \quad \mathcal{U}y = \frac{y}{\sqrt{\bar{\mu}}}$$

- **Spectral properties:** $\mathcal{H}$ has discrete spectrum and only **finitely many** unstable modes $(\Re(\lambda) \geq -\delta)$[11]

- **Mass conservation:** $\int_\Omega \mathcal{L}y\,dx = 0$, hence 0 is a neutral eigenvalue.
- **Work on** $\mathcal{X}_0 := \{y \in L^2(\Omega, \bar{\mu}^{-1}) : \int_\Omega y\,dx = 0\}$
- The proof relies on this spectral structure, which can be extended to (more) general **Wasserstein gradient flows**.

[10] Kalise, Moschen, Pavliotis (2025).

[11] Reed, Simon (1978).

Structure of the Closed-Loop Error Equation

McKean–Vlasov closed loop in transformed variables

- **Feedback-modified free energy:**

$$\mathcal{F}_{\mathsf{fb}}(\mu, t) = \mathcal{F}(\mu) + \sum_{j=1}^{m} u_j(t) \int_{\Omega} \alpha_j \, d\mu, \qquad u(t) = -\mathcal{B}^* \Pi \left(\mu_t - \bar{\mu}\right)$$

- **Nonlinear remainder:** satisfies a local Lipschitz estimate
- **Transformed equation:**

$$\dot{z} = -\mathcal{H}_{\Pi} z - \left(\hat{\mathcal{B}}^* \hat{\Pi} z\right) \hat{\mathcal{N}}^{\delta}(t) z + \hat{\mathcal{W}}^{\delta}(z), \quad z(0, \cdot) = \mathcal{U}(\mu_0 - \bar{\mu})$$

where $\mathcal{H}_{\Pi} = \mathcal{H} - \delta I + \hat{\mathcal{B}}\hat{\mathcal{B}}^* \hat{\Pi}$.

- **Notation:** $z = e^{\delta t} \mathcal{U} y$, $\hat{\mathcal{B}} = \mathcal{U}\mathcal{B}$, $\hat{\Pi} = \mathcal{U}\Pi\mathcal{U}^{-1}$.

$$\hat{\mathcal{N}}_j^{\delta}(t) z := e^{-\delta t} \mathcal{U} \mathcal{N}_j \mathcal{U}^{-1} z, \quad \hat{\mathcal{W}}^{\delta}(z) := e^{-2\delta t} \mathcal{U} \mathcal{W}(\mathcal{U}^{-1} z),$$

with $\mathcal{N}_j y = \nabla \cdot (y \nabla \alpha_j)$ and $\mathcal{W}(y) = \nabla \cdot (y \nabla W * y)$.

Main Result: Local Exponential Stabilisation

- **Theorem.** There exist $\varepsilon > 0$ and C such that if $\|\mu_0 - \bar{\mu}\|_{L^2(\bar{\mu}^{-1})} \leq \varepsilon$, the closed-loop McKean-Vlasov equation admits a unique solution and

$$\|\mu_t - \bar{\mu}\|_{L^2(\bar{\mu}^{-1})} \leq Ce^{-\delta t} \|\mu_0 - \bar{\mu}\|_{L^2(\bar{\mu}^{-1})}$$

- **Proof Strategy (Fixed Point):**
  1. **Linear Stability:** The operator $\mathcal{H}_\Pi$ generates an exponentially stable semigroup $e^{-\mathcal{H}_\Pi t}$. This leads to the exponential decay with rate δ
  2. **Nonlinear Estimate:** The Lipschitz property of the nonlinear remainder allows us to close the contraction loop for small initial data.
  3. **Result:** The full nonlinear system contracts in the transformed space $L^2(\Omega)$.

Numerical implementation

Work on the periodic domain $\Omega = \mathbb{T}^d$

- **Fourier expansion:** for $k \in \mathbb{Z}^d$,

$$y(x,t) \approx \frac{1}{\sqrt{(2\pi)^d}} \sum_{|k| \le N} \hat{y}_k(t) e^{ik\cdot x}$$

- **Galerkin projection:** project the full equation onto the span of $\left\{ \frac{1}{\sqrt{(2\pi)^d}} e^{ik\cdot x} \right\}_{|k|\le N}$. The matrix representation is

$$\dot{\hat{y}} = L\hat{y} + \sum_{j=1}^{m} B_j u_j, \qquad J = \int_0^\infty e^{2\delta t} (\hat{y}^\top M \hat{y} + \|u\|^2)\, dt$$

- **Feedback computation:** solve the discrete Riccati equation

$$(L^\top + \delta I)\Pi_N + \Pi_N(L + \delta I) - \Pi_N B B^\top \Pi_N + M = 0, \quad u = -B^\top \Pi_N \hat{y}$$

Noisy Kuramoto under feedback control

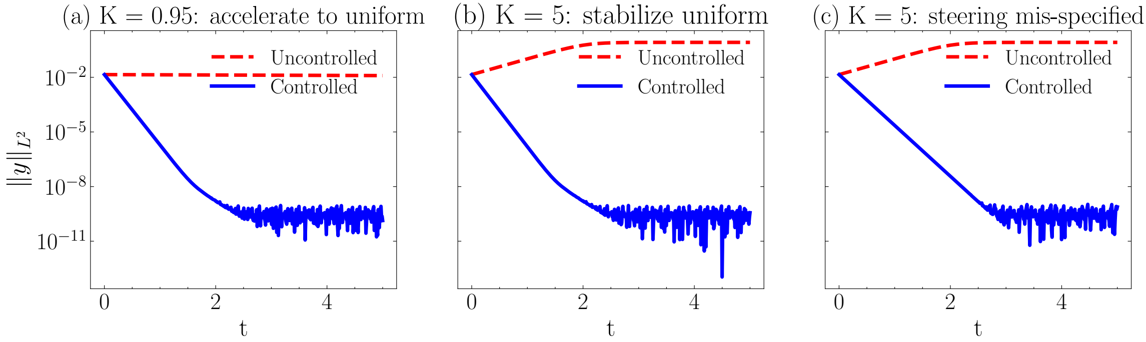


Figure: Set $V(x) = 0$, $W(x) = -K\cos(x)$ and $\sigma = 0.5$.

Effect of the Coupling Strength K under Feedback

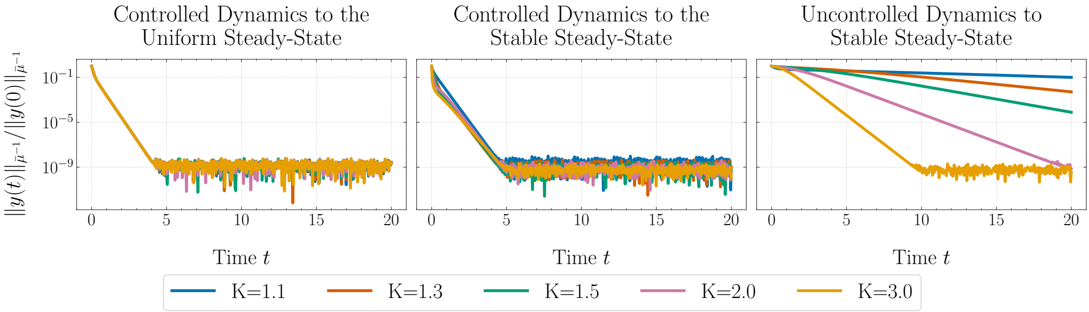


Figure: Evolution of $\|y(t)\|_{\bar{\mu}^{-1}}/\|y(0)\|_{\bar{\mu}^{-1}}$ for different coupling strengths K. The same feedback law enforces exponential decay with prescribed rate $\delta = 3$.

Kuramoto model + Potential V

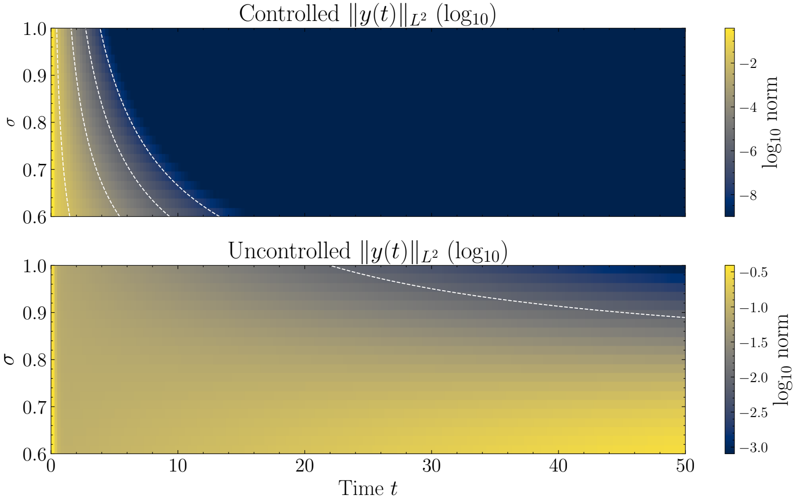


Figure: Set $V(x) = 0.05\cos(x)$ and $W(x) = -\cos(x)$

Particle-Level Consequences of the Feedback Law

- **Closed-loop particle system.** The feedback control induces the interacting SDE

$$dX_t^{i,u} = -\nabla\Big(V(X_t^{i,u}) + (W * \mu_t^{N,u})(X_t^{i,u}) + \sum_{j=1}^{m} u_j(t)\alpha_j(X_t^{i,u})\Big)\,dt + \sqrt{2\sigma}\,dB_t^i$$

- **Well-posedness.** For each N, the controlled system admits a unique strong solution.

- **Propagation of chaos under feedback.** The empirical law converges to the mean-field law[12]

$$\sup_{t\le T} \mathbb{E}\,\mathsf{W}_2^2(\mu_t^{N,u}, \mu_t) \le C_T\,\beta_d(N), \qquad \beta_d(N) \to 0.$$

- **Uniform convergence to equilibrium.** Combining propagation of chaos with exponential stability of the mean field,

$$\mathbb{E}\,\mathsf{W}_2^2(\mu_t^{N,u}, \bar{\mu}) \le C\,\beta_d(N) + Ce^{-2\delta t}.$$

[12] Fournier, Guillin (2015).

Conclusions and Current Frontiers

Conclusions

- Feedback local stabilisation with prescribed exponential rate for McKean-Vlasov PDEs on $\mathbb{T}^d$
- Linearisation in a Hilbert setting using Otto calculus
- Reduction of nonlinear PDE stabilisation to identifying finitely many unstable spectral modes

Current Frontiers

- Extension to $\mathbb{R}^d$ under confining potentials with explicit rates
- Local convexification of the free energy $\mathcal{F}$
- Control of mean-field and kinetic models beyond gradient-flow structure

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Thank you for your attention!

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\psi_n(x), \quad \mathcal{L}\psi_n = \lambda_n \psi_n, \quad \langle \psi_n^*, \psi_m \rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t) \langle \mathcal{B}_j, \psi_n^* \rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j \coloneqq \nabla \cdot (\bar{\mu} \nabla \alpha_j) = \psi_j, \quad j = 1, \ldots, m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j) \geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1, \ldots, m$$

$\mathcal{B}_j = \psi_j \Rightarrow$ *infinite-dimensional Hautus condition* $\Rightarrow$ δ*-stability of the linearised system*