IMPERIAL

16th Viennese Conference on Optimal Control and Dynamic Games

Feedback stabilisation for the McKean-Vlasov equation

Controlling the long-time behavior of interacting particle systems

Dante Kalise, **Lucas M. Moschen**, Greg Pavliotis

This project is supported by the CNRS-ICL PhD program.

July 17th, 2025

Ceilidh Dance
From local rules to global behaviour

- A traditional Scottish dance with a caller who steers the choreography (pattern).
- Local interactions lead to global co-ordination
- We model the agents with stochastic differential equations (SDEs)
- In the limit of many agents, the probability density follows a PDE: the **McKean-Vlasov equation**

The McKean-Vlasov Equation[1][2]

$$\partial_t \mu = \underbrace{\sigma \Delta \mu}_{\text{Diffusion}} + \underbrace{\nabla \cdot (\mu \nabla V)}_{\text{External drift}} + \underbrace{\nabla \cdot (\mu \nabla W * \mu)}_{\text{Interaction drift}}$$

- $\mu(t, x)$: evolution of the density of agents
- $W(x)$: introduces **nonlinearity** and **nonlocality** through $(\nabla W * y)(x) := \int_\Omega \nabla W(x - x')\, y(x')\, dx'$
- Fokker–Planck PDE for the McKean–Vlasov SDE where the drift depends on the law μ

[1] A.-S. Sznitman, Topics in propagation of chaos, Lecture Notes in Mathematics, vol. 1464, Springer, 1991.
[2] J.A. Carrillo, R.J. McCann, and C. Villani, Kinetic equilibration rates for granular media and related equations, Rev. Mat. Iberoamericana, 2003.

Why Control Interacting Systems?
Motivations for feedback design

- Uncontrolled systems:
  - Possible **slow convergence** to the long-term (small spectral gap)
  - Converge to an **undesirable equilibrium**
- Our goals:
  - **Accelerate convergence** to a desired steady state
  - **Steer the system** toward or away from specific modes
- Solution approach:
  - Introduce **time-dependent feedback potentials** into the PDE dynamics
  - Based on Breiten, Kunisch, and Pfeiffer's work[3]

Why Control Interacting Systems?
Motivations for feedback design

- Uncontrolled systems:
  - Possible **slow convergence** to the long-term (small spectral gap)
  - Converge to an **undesirable equilibrium**

- Our goals:
  - **Accelerate convergence** to a desired steady state
  - **Steer the system** toward or away from specific modes

- Solution approach:
  - Introduce **time-dependent feedback potentials** into the PDE dynamics
  - Based on Breiten, Kunisch, and Pfeiffer's work[3]

Why Control Interacting Systems?
Motivations for feedback design

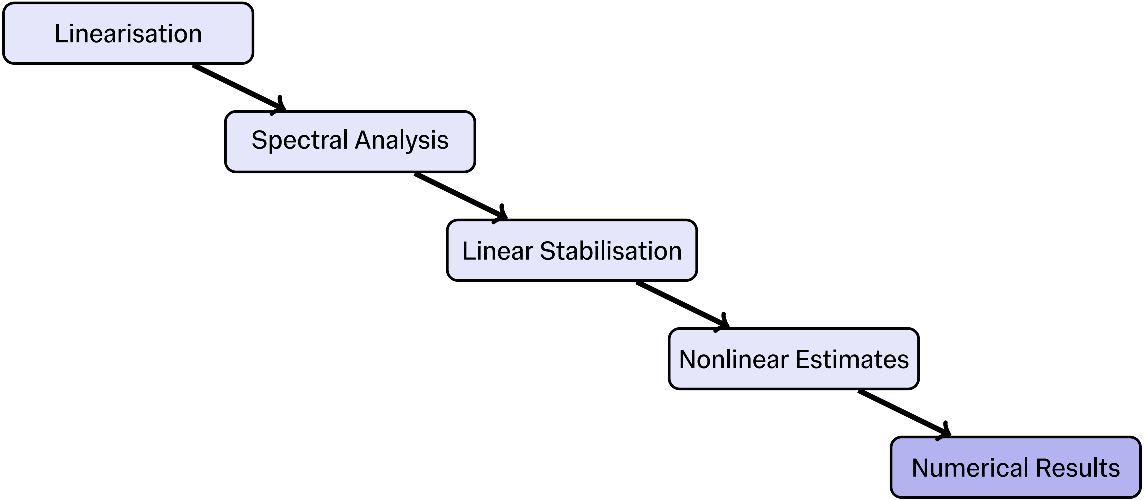
- Uncontrolled systems:
  - Possible **slow convergence** to the long-term (small spectral gap)
  - Converge to an **undesirable equilibrium**

- Our goals:
  - **Accelerate convergence** to a desired steady state
  - **Steer the system** toward or away from specific modes

- Solution approach:
  - Introduce **time-dependent feedback potentials** into the PDE dynamics
  - Based on Breiten, Kunisch, and Pfeiffer's work[3]

[3] T. Breiten, K. Kunisch, and L. Pfeiffer. Control strategies for the Fokker-Planck equation. ESAIM: COCV, 2018.

Overview of Our Approach



From Interacting Particles to the Mean-Field PDE

- **Agent-based model**: each of N agents evolve according to

$$dX_i(t) = -\nabla V(X_i)\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_i - X_j)\,dt + \sqrt{2\sigma}\,dB_i(t)$$

- **Mean-field limit** $N \to \infty$: overdamped regime

$$\partial_t \mu = \nabla \cdot (\sigma \nabla \mu + \mu(\nabla V + \nabla W * \mu)), \quad t > 0,\ x \in \Omega$$

- **Conditions**:
  - We consider $\Omega = \mathbb{R}^n$, with decay, or $\Omega = \mathbb{T}^n$, with periodic boundary conditions
  - We ask $W \in W^{2,\infty}(\Omega)$ and $V \in C^2(\Omega)$. If $\Omega = \mathbb{R}^n$,

$$\lim_{|x|\to\infty} \frac{1}{4\sigma}|\nabla V(x)|^2 - \frac{1}{2}\Delta V(x) = \infty$$

From Interacting Particles to the Mean-Field PDE

- **Agent-based model**: each of N agents evolve according to

$$dX_i(t) = -\nabla V(X_i)\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_i - X_j)\,dt + \sqrt{2\sigma}\,dB_i(t)$$

- **Mean-field limit** $N \to \infty$: overdamped regime

$$\partial_t \mu = \nabla \cdot (\sigma \nabla \mu + \mu(\nabla V + \nabla W * \mu))\,, \quad t > 0,\, x \in \Omega$$

- **Conditions**:
  - We consider $\Omega = \mathbb{R}^n$, with decay, or $\Omega = \mathbb{T}^n$, with periodic boundary conditions
  - We ask $W \in W^{2,\infty}(\Omega)$ and $V \in C^2(\Omega)$. If $\Omega = \mathbb{R}^n$,

$$\lim_{|x|\to\infty} \frac{1}{4\sigma}|\nabla V(x)|^2 - \frac{1}{2}\Delta V(x) = \infty$$

From Interacting Particles to the Mean-Field PDE

- **Agent-based model**: each of N agents evolve according to

$$dX_i(t) = -\nabla V(X_i)\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla W(X_i - X_j)\,dt + \sqrt{2\sigma}\,dB_i(t)$$

- **Mean-field limit** $N \to \infty$: overdamped regime

$$\partial_t \mu = \nabla \cdot (\sigma \nabla \mu + \mu(\nabla V + \nabla W * \mu))\,, \quad t > 0,\ x \in \Omega$$

- **Conditions**:
  - We consider $\Omega = \mathbb{R}^n$, with decay, or $\Omega = \mathbb{T}^n$, with periodic boundary conditions
  - We ask $W \in W^{2,\infty}(\Omega)$ and $V \in C^2(\Omega)$. If $\Omega = \mathbb{R}^n$,

$$\lim_{|x|\to\infty} \frac{1}{4\sigma}|\nabla V(x)|^2 - \frac{1}{2}\Delta V(x) = \infty$$

Long-Time Behaviour of the McKean-Vlasov Equation

- **Steady states:** time-independent solutions $\bar{\mu}$ satisfying

$$\nabla \cdot (\sigma \nabla \bar{\mu} + \bar{\mu}(\nabla \mathsf{V} + \nabla \mathsf{W} * \bar{\mu})) = 0 \quad \Longrightarrow \quad \bar{\mu} \propto \exp\left\{-\frac{1}{\sigma}(\mathsf{V} + \mathsf{W} * \bar{\mu})\right\}$$

- **Gradient flow:** the dynamics decrease a free energy functional. Steady states are critical points of this energy

- **Convex potentials:** convexity of V and W implies uniqueness of $\bar{\mu}$ and $\mu(t, \cdot) \xrightarrow{L_1} \bar{\mu}$ exponentially[4]

- **Nonconvex potentials:** multiple steady states may emerge.

Long-Time Behaviour of the McKean-Vlasov Equation

- **Steady states:** time-independent solutions $\bar{\mu}$ satisfying

$$\nabla \cdot (\sigma \nabla \bar{\mu} + \bar{\mu}(\nabla \mathsf{V} + \nabla \mathsf{W} * \bar{\mu})) = 0 \quad \Longrightarrow \quad \bar{\mu} \propto \exp\left\{ -\frac{1}{\sigma}(\mathsf{V} + \mathsf{W} * \bar{\mu}) \right\}$$

- **Gradient flow:** the dynamics decrease a free energy functional. Steady states are critical points of this energy
- **Convex potentials:** convexity of V and W implies uniqueness of $\bar{\mu}$ and $\mu(\mathrm{t}, \cdot) \xrightarrow{\mathrm{L}^1} \bar{\mu}$ exponentially[4]
- **Nonconvex potentials:** multiple steady states may emerge.

Long-Time Behaviour of the McKean-Vlasov Equation

- **Steady states:** time-independent solutions $\bar{\mu}$ satisfying

$$\nabla \cdot (\sigma \nabla \bar{\mu} + \bar{\mu}(\nabla \mathsf{V} + \nabla \mathsf{W} * \bar{\mu})) = 0 \quad \Longrightarrow \quad \bar{\mu} \propto \exp\left\{ -\frac{1}{\sigma}(\mathsf{V} + \mathsf{W} * \bar{\mu}) \right\}$$

- **Gradient flow:** the dynamics decrease a free energy functional. Steady states are critical points of this energy

- **Convex potentials:** convexity of V and W implies uniqueness of $\bar{\mu}$ and $\mu(\mathsf{t}, \cdot) \xrightarrow{\mathsf{L}_1} \bar{\mu}$ exponentially[4]

- **Nonconvex potentials:** multiple steady states may emerge.

[4] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDEs. Stoch. Proc. Appl, 2001.

Long-Time Behaviour of the McKean-Vlasov Equation

- **Steady states:** time-independent solutions $\bar{\mu}$ satisfying

$$\nabla \cdot (\sigma \nabla \bar{\mu} + \bar{\mu}(\nabla \mathsf{V} + \nabla \mathsf{W} * \bar{\mu})) = 0 \quad \Longrightarrow \quad \bar{\mu} \propto \exp\left\{-\frac{1}{\sigma}(\mathsf{V} + \mathsf{W} * \bar{\mu})\right\}$$

- **Gradient flow:** the dynamics decrease a free energy functional. Steady states are critical points of this energy

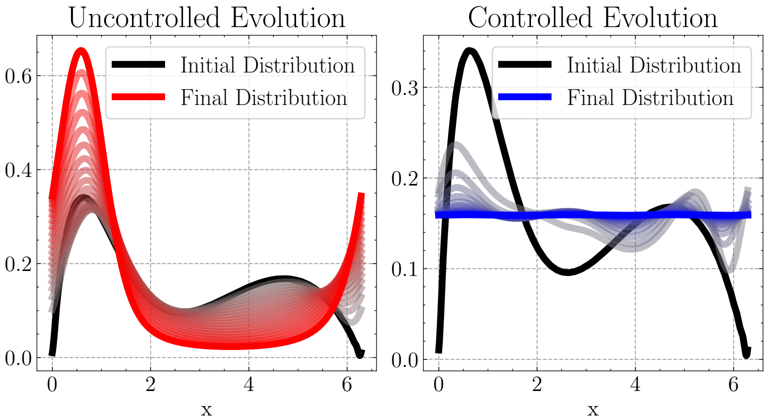
- **Convex potentials:** convexity of V and W implies uniqueness of $\bar{\mu}$ and $\mu(\mathsf{t}, \cdot) \xrightarrow{\mathsf{L}_1} \bar{\mu}$ exponentially[4]

- **Nonconvex potentials:** multiple steady states may emerge.

[4] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDEs. Stoch. Proc. Appl, 2001.

Noisy Kuramoto Model
What kind of problem?

- Coupled phase oscillators with sine interaction that synchronize[5]
- Let $\Omega = [0, 2\pi)$, $\mathsf{V}(\mathsf{x}) = 0$, $\mathsf{W}(\mathsf{x}) = -\mathsf{K}\cos(\mathsf{x})$
- If $\mathsf{K} \leq 1$: $\bar{\mu}(\mathsf{x}) = \frac{1}{2\pi}$ is the unique steady state
- If $\mathsf{K} > 1$: $\bar{\mu} = \frac{1}{2\pi}$ becomes unstable and infinite new steady states appear



[5] L. Bertini, G. Giacomin, K. Pakdaman. Dynamical Aspects of Mean Field Plane Rotators and the Kuramoto Model, J Stat Phys. 2010.

Formulation of the Controlled Dynamics

- **Objective:** accelerate convergence to $\bar{\mu}$ or stabilise otherwise unstable equilibria

- **Control strategy:** modify the potential as[6]

$$V(x) \mapsto V(x) + \overbrace{\sum_{j=1}^{m} u_j(t)\,\alpha_j(x)}^{\text{Control input}},$$

  for **chosen** spatial controls α_j and **to be optimized** time-dependent controls u_j

- **Controlled PDE:**

$$\partial_t \mu = \nabla \cdot \left(\mu \left(\nabla V + \nabla W * \mu + \sum_{j=1}^{m} u_j \nabla \alpha_j \right) \right) + \sigma \Delta \mu$$

[6] T. Breiten, K. Kunisch, and L. Pfeiffer. Control strategies for the Fokker-Planck equation. ESAIM: COCV, 2018.

Operator-Theoretic Reformulation

- **Rewriting the equation:** let $y = \mu - \bar{\mu}$, then

$$\dot{y} = \overbrace{(\mathcal{A} + \mathcal{D}_W)y}^{\text{linear part}} + \sum_{j=1}^{m} \overbrace{u_j(t)\,\mathcal{N}_j y}^{\text{bilinear control}} + \sum_{j=1}^{m} \overbrace{\mathcal{B}_j\, u_j(t)}^{\text{pure input term}} + \overbrace{\mathcal{W}(y),}^{\text{nonlinear remainder}} \qquad \mathcal{B}_j = \mathcal{N}_j \bar{\mu}$$

- **Operators:** $\mathcal{A}\mu = \nabla \cdot (\sigma \nabla \mu + \mu \nabla V)$, $\quad \mathcal{N}_j \mu = \nabla \cdot (\mu \nabla \alpha_j)$, $\quad \mathcal{W}(\mu) = \nabla \cdot (\mu \nabla W * \mu)$

- **State space:** $\mathcal{X} = L^2(\Omega, \bar{\mu}^{-1}) := \left\{ f : \int_\Omega |f(x)|^2 \bar{\mu}^{-1}(x)\, dx < \infty \right\}$

- **Linearization:** the Fréchet derivative $\mathcal{D}_W$ of $\mathcal{W}$ at $\bar{\mu}$ is

$$\mathcal{W}(\bar{\mu} + y) = \underbrace{\nabla \cdot \left[\overbrace{\bar{\mu}(\nabla W * y)}^{\text{nonlocal part } \mathcal{D}_{W,2}} + \overbrace{y(\nabla W * \bar{\mu})}^{\text{local part } \mathcal{D}_{W,1}} \right]}_{\mathcal{D}_W y} + \underbrace{\nabla \cdot [y(\nabla W * y)]}_{\mathcal{W}(y)}$$

How do we plan to control?

- **Goal:** Stabilise an equilibrium $\bar{\mu}$ of the McKean–Vlasov equation via **feedback control**.

- **Linearised dynamics around** $\bar{\mu}$:

$$\dot{y} = (\mathcal{A} + \mathcal{D}_W)y + \cancel{\sum_{j=1}^{m} u_j(t)\mathcal{N}_j y} + \sum_{j=1}^{m} \mathcal{B}_j u_j(t) + \cancel{\mathcal{W}(y)},$$

where $\mathcal{B}_j = \nabla \cdot (\bar{\mu}\nabla\alpha_j)$. Fix $\delta > 0$. We **have to** choose α_j.

- **Key idea:** control the **directions** related to eigenvalues with real part $\geq -\delta$.

- **Strategy:**

Spectral analysis of $\mathcal{A} + \mathcal{D}_W$ → Identify unstable modes ($\Re\lambda \geq -\delta$) → Project and build a finite-dim feedback → Closed-loop stability Contraction argument

How do we plan to control?

- **Goal:** Stabilise an equilibrium $\bar{\mu}$ of the McKean–Vlasov equation via **feedback control**.

- **Linearised dynamics around** $\bar{\mu}$:

$$\dot{y} = (\mathcal{A} + \mathcal{D}_W)y + \cancel{\sum_{j=1}^{m} u_j(t)\mathcal{N}_j y} + \sum_{j=1}^{m} \mathcal{B}_j u_j(t) + \cancel{\mathcal{W}(y)},$$

where $\mathcal{B}_j = \nabla \cdot (\bar{\mu}\nabla\alpha_j)$. Fix $\delta > 0$. We **have to** choose α_j.

- **Key idea:** control the **directions** related to eigenvalues with real part $\geq -\delta$.

- **Strategy:**

Spectral analysis of $\mathcal{A} + \mathcal{D}_W$ → Identify unstable modes ($\Re\lambda \geq -\delta$) → Project and build a finite-dim feedback → Closed-loop stability Contraction argument

How do we plan to control?

- **Goal:** Stabilise an equilibrium $\bar{\mu}$ of the McKean–Vlasov equation via **feedback control**.

- **Linearised dynamics around** $\bar{\mu}$:

$$\dot{y} = (\mathcal{A} + \mathcal{D}_W)y + \cancel{\textstyle\sum_{j=1}^{m} u_j(t)\mathcal{N}_j y} + \sum_{j=1}^{m} \mathcal{B}_j u_j(t) + \cancel{\mathcal{W}(y)},$$

  where $\mathcal{B}_j = \nabla \cdot (\bar{\mu}\nabla\alpha_j)$. Fix $\delta > 0$. We **have to** choose α_j.

- **Key idea:** control the **directions** related to eigenvalues with real part $\geq -\delta$.

- **Strategy:**

  Spectral analysis of $\mathcal{A} + \mathcal{D}_W$ → Identify unstable modes ($\Re\lambda \geq -\delta$) → Project and build a finite-dim feedback → Closed-loop stability Contraction argument

How do we plan to control?

- **Goal:** Stabilise an equilibrium $\bar{\mu}$ of the McKean–Vlasov equation via **feedback control**.

- **Linearised dynamics around** $\bar{\mu}$:

$$\dot{y} = (\mathcal{A} + \mathcal{D}_W)y + \cancel{\sum_{j=1}^{m} u_j(t)\mathcal{N}_j y} + \sum_{j=1}^{m} \mathcal{B}_j u_j(t) + \cancel{\mathcal{W}(y)},$$

  where $\mathcal{B}_j = \nabla \cdot (\bar{\mu}\nabla\alpha_j)$. Fix $\delta > 0$. We **have to** choose α_j.

- **Key idea:** control the **directions** related to eigenvalues with real part $\geq -\delta$.

- **Strategy:**

Spectral analysis of $\mathcal{A} + \mathcal{D}_W$ → Identify unstable modes ($\Re\lambda \geq -\delta$) → Project and build a finite-dim feedback → Closed-loop stability Contraction argument

Spectral Reformulation via Unitary Transformation

- **Ground-state transform:** the unitary map

$$\mathcal{U} : \mathsf{L}^2(\Omega, \bar{\mu}^{-1}) \to \mathsf{L}^2(\Omega), \quad \mathcal{U}\mathsf{y} = \frac{\mathsf{y}}{\sqrt{\bar{\mu}}}$$

- **Transformed operator:** the linearised operator becomes

$$\mathcal{H} := -\mathcal{U}(\mathcal{A} + \mathcal{D}_{\mathsf{W}})\mathcal{U}^{-1} = \overbrace{-\sigma\Delta + \underbrace{\Psi(\mathsf{x})}_{\Psi = \frac{1}{4\sigma}|\nabla \mathsf{V}|^2 - \frac{1}{2}\Delta \mathsf{V}}}^{\text{Schrödinger operator}} + \overbrace{\mathcal{K}}^{\text{Hilbert–Schmidt operator}}$$

- **Spectral structure:**
  - Rich spectral theory and numerical methods for Schrödinger operators
  - $\mathcal{H}$ has compact resolvent and a pure point spectrum[7]
  - $\sigma(\mathcal{H}) = -\sigma(\mathcal{A} + \mathcal{D}_{\mathsf{W}})$, with isolated eigenvalues **accumulating at infinity**
  - There are **finitely many eigenvalues** with $\Re(\lambda) \geq -\delta \longrightarrow$ we control these modes

Spectral Reformulation via Unitary Transformation

- **Ground-state transform:** the unitary map

$$\mathcal{U} : \mathsf{L}^2(\Omega, \bar{\mu}^{-1}) \to \mathsf{L}^2(\Omega), \quad \mathcal{U}\mathsf{y} = \frac{\mathsf{y}}{\sqrt{\bar{\mu}}}$$

- **Transformed operator:** the linearised operator becomes

$$\mathcal{H} := -\mathcal{U}(\mathcal{A} + \mathcal{D}_{\mathsf{W}})\mathcal{U}^{-1} = \overbrace{-\sigma\Delta + \underbrace{\Psi(\mathsf{x})}_{\Psi = \frac{1}{4\sigma}|\nabla \mathsf{V}|^2 - \frac{1}{2}\Delta \mathsf{V}}}^{\text{Schrödinger operator}} + \overbrace{\mathcal{K}}^{\text{Hilbert–Schmidt operator}}$$

- **Spectral structure:**
  - Rich spectral theory and numerical methods for Schrödinger operators
  - $\mathcal{H}$ has compact resolvent and a pure point spectrum[7]
  - $\sigma(\mathcal{H}) = -\sigma(\mathcal{A} + \mathcal{D}_{\mathsf{W}})$, with isolated eigenvalues **accumulating at infinity**
  - There are **finitely many eigenvalues** with $\Re(\lambda) \geq -\delta \longrightarrow$ we control these modes

Spectral Reformulation via Unitary Transformation

- **Ground-state transform:** the unitary map

$$\mathcal{U} : \mathsf{L}^2(\Omega, \bar{\mu}^{-1}) \to \mathsf{L}^2(\Omega), \quad \mathcal{U}\mathsf{y} = \frac{\mathsf{y}}{\sqrt{\bar{\mu}}}$$

- **Transformed operator:** the linearised operator becomes

$$\mathcal{H} := -\mathcal{U}(\mathcal{A} + \mathcal{D}_{\mathsf{W}})\mathcal{U}^{-1} = \overbrace{-\sigma\Delta + \underbrace{\Psi(\mathsf{x})}_{\Psi = \frac{1}{4\sigma}|\nabla \mathsf{V}|^2 - \frac{1}{2}\Delta \mathsf{V}}}^{\text{Schrödinger operator}} + \overbrace{\mathcal{K}}^{\text{Hilbert–Schmidt operator}}$$

- **Spectral structure:**
  - Rich spectral theory and numerical methods for Schrödinger operators
  - $\mathcal{H}$ has compact resolvent and a pure point spectrum[7]
  - $\sigma(\mathcal{H}) = -\sigma(\mathcal{A} + \mathcal{D}_{\mathsf{W}})$, with isolated eigenvalues **accumulating at infinity**
  - There are **finitely many eigenvalues** with $\Re(\lambda) \geq -\delta \longrightarrow$ we control these modes

[7] M. Reed and B. Simon. Analysis of Operators; Methods of Modern Mathematical Physics IV. 1978.

Mass Conservation and Zero-Mean Projection

- **Observation:** the operator $\mathcal{A} + \mathcal{D}_{\mathsf{W}}$ preserves mass:

$$\int_\Omega (\mathcal{A} + \mathcal{D}_{\mathsf{W}})\mathsf{y}(\mathsf{x})\,\mathsf{dx} = \int_\Omega \nabla \cdot \left(\sigma \nabla \mathsf{y}(\mathsf{x}) + \mathsf{y}(\mathsf{x}) \nabla (\mathsf{V} + \mathsf{W} * \bar{\mu})(\mathsf{x}) + \bar{\mu}(\mathsf{x})(\nabla \mathsf{W} * \mathsf{y})(\mathsf{x})\right) \mathsf{dx} = 0,$$

so $0 = \langle (\mathcal{A} + \mathcal{D}_{\mathsf{W}})\mathsf{y}, \bar{\mu} \rangle_{\mathsf{L}^2(\Omega, \bar{\mu}^{-1})}$ for all $\mathsf{y} \in \mathsf{L}^2(\Omega, \bar{\mu}^{-1})$

- **Consequence:** $(\mathcal{A} + \mathcal{D}_{\mathsf{W}})^* \bar{\mu} = 0 \implies 0$ is an eigenvalue of $\mathcal{A} + \mathcal{D}_{\mathsf{W}}$ we don't need to control

- **Fix:** we work in the subspace of zero-mean perturbations:

$$\mathcal{X}_0 := \left\{ \mathsf{y} \in \mathsf{L}^2(\Omega, \bar{\mu}^{-1}) : \int_\Omega \mathsf{y}(\mathsf{x})\,\mathsf{dx} = 0 \right\}$$

- **Notation:** for presentation purposes, we keep writing $\mathcal{A}$ and $\mathcal{D}_{\mathsf{W}}$ but implicitly restrict them to $\mathcal{X}_0$

Mass Conservation and Zero-Mean Projection

- **Observation:** the operator $\mathcal{A} + \mathcal{D}_W$ preserves mass:

$$\int_\Omega (\mathcal{A} + \mathcal{D}_W) y(x) \, dx = \int_\Omega \nabla \cdot \left(\sigma \nabla y(x) + y(x) \nabla (V + W * \bar{\mu})(x) + \bar{\mu}(x)(\nabla W * y)(x)\right) dx = 0,$$

  so $0 = \langle (\mathcal{A} + \mathcal{D}_W) y, \bar{\mu} \rangle_{L^2(\Omega, \bar{\mu}^{-1})}$ for all $y \in L^2(\Omega, \bar{\mu}^{-1})$

- **Consequence:** $(\mathcal{A} + \mathcal{D}_W)^* \bar{\mu} = 0 \implies 0$ is an eigenvalue of $\mathcal{A} + \mathcal{D}_W$ we don't need to control

- **Fix:** we work in the subspace of zero-mean perturbations:

$$\mathcal{X}_0 := \left\{ y \in L^2(\Omega, \bar{\mu}^{-1}) : \int_\Omega y(x) \, dx = 0 \right\}$$

- **Notation:** for presentation purposes, we keep writing $\mathcal{A}$ and $\mathcal{D}_W$ but implicitly restrict them to $\mathcal{X}_0$

Mass Conservation and Zero-Mean Projection

- **Observation:** the operator $\mathcal{A} + \mathcal{D}_\mathsf{W}$ preserves mass:

$$\int_\Omega (\mathcal{A} + \mathcal{D}_\mathsf{W})\mathsf{y}(\mathsf{x})\,\mathsf{dx} = \int_\Omega \nabla \cdot (\sigma \nabla \mathsf{y}(\mathsf{x}) + \mathsf{y}(\mathsf{x})\nabla(\mathsf{V} + \mathsf{W} * \bar{\mu})(\mathsf{x}) + \bar{\mu}(\mathsf{x})(\nabla \mathsf{W} * \mathsf{y})(\mathsf{x}))\,\mathsf{dx} = 0,$$

  so $0 = \langle (\mathcal{A} + \mathcal{D}_\mathsf{W})\mathsf{y}, \bar{\mu}\rangle_{\mathsf{L}^2(\Omega,\bar{\mu}^{-1})}$ for all $\mathsf{y} \in \mathsf{L}^2(\Omega, \bar{\mu}^{-1})$

- **Consequence:** $(\mathcal{A} + \mathcal{D}_\mathsf{W})^* \bar{\mu} = 0 \implies 0$ is an eigenvalue of $\mathcal{A} + \mathcal{D}_\mathsf{W}$ we don't need to control

- **Fix:** we work in the subspace of zero-mean perturbations:

$$\mathcal{X}_0 := \left\{ \mathsf{y} \in \mathsf{L}^2(\Omega, \bar{\mu}^{-1}) : \int_\Omega \mathsf{y}(\mathsf{x})\,\mathsf{dx} = 0 \right\}$$

- **Notation:** for presentation purposes, we keep writing $\mathcal{A}$ and $\mathcal{D}_\mathsf{W}$ but implicitly restrict them to $\mathcal{X}_0$

Optimal Control and Stabilisation of the Linearised Problem

Consider the problem

$$\min_{u(\cdot)} \frac{1}{2}\int_0^\infty e^{2\delta t}\left(\left\langle y(t), \mathcal{M}\, y(t)\right\rangle + \|u(t)\|^2\right) dt, \quad \mathcal{M} \text{ positive definite}$$

$$\text{subject to} \quad \dot{y}(t) = \underbrace{(\mathcal{A} + \mathcal{D}_W)}_{=:\mathcal{L}} y(t) + \underbrace{[\mathcal{B}_1, \ldots, \mathcal{B}_m]}_{=:\mathcal{B}} u(t),$$

- **Riccati equation:** find a self-adjoint operator Π solving

$$(\mathcal{L}^* + \delta I)\Pi + \Pi(\mathcal{L} + \delta I) - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{M} = 0$$

- **Feedback law:** $u(t) = -\mathcal{B}^*\Pi y(t)$ stabilises the system, i.e., $\|y\|_{L^2(\Omega,\bar{\mu}^{-1})} \le Ce^{-\delta t}$. We need to verify the **infinite-dimensional Hautus condition**[8]

Optimal Control and Stabilisation of the Linearised Problem

Consider the problem

$$\min_{u(\cdot)} \frac{1}{2}\int_0^\infty e^{2\delta t}\left(\left\langle y(t), \mathcal{M}\, y(t)\right\rangle + \|u(t)\|^2\right) dt, \quad \mathcal{M} \text{ positive definite}$$

$$\text{subject to} \quad \dot{y}(t) = \underbrace{(\mathcal{A} + \mathcal{D}_W)}_{=:\mathcal{L}}\, y(t) + \underbrace{[\mathcal{B}_1, \ldots, \mathcal{B}_m]}_{=:\mathcal{B}}\, u(t),$$

- **Riccati equation:** find a self-adjoint operator Π solving

$$(\mathcal{L}^* + \delta I)\Pi + \Pi(\mathcal{L} + \delta I) - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{M} = 0$$

- **Feedback law:** $u(t) = -\mathcal{B}^*\Pi y(t)$ stabilises the system, i.e., $\|y\|_{L^2(\Omega,\bar{\mu}^{-1})} \le Ce^{-\delta t}$. We need to verify the **infinite-dimensional Hautus condition**[8]

Optimal Control and Stabilisation of the Linearised Problem

Consider the problem

$$\min_{u(\cdot)} \frac{1}{2}\int_0^\infty e^{2\delta t}\left(\left\langle y(t), \mathcal{M}\, y(t)\right\rangle + \|u(t)\|^2\right) dt, \quad \mathcal{M} \text{ positive definite}$$

$$\text{subject to} \quad \dot{y}(t) = \underbrace{(\mathcal{A} + \mathcal{D}_W)}_{=:\mathcal{L}}\, y(t) + \underbrace{\left[\mathcal{B}_1, \ldots, \mathcal{B}_m\right]}_{=:\mathcal{B}}\, u(t),$$

- **Riccati equation:** find a self-adjoint operator Π solving

$$(\mathcal{L}^* + \delta I)\Pi + \Pi(\mathcal{L} + \delta I) - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{M} = 0$$

- **Feedback law:** $u(t) = -\mathcal{B}^*\Pi y(t)$ stabilises the system, i.e., $\|y\|_{L^2(\Omega,\bar{\mu}^{-1})} \leq Ce^{-\delta t}$. We need to verify the **infinite-dimensional Hautus condition**[8]

[8] R. Curtain and H. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, 2005.

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\varphi_n(x), \quad (\mathcal{A}+\mathcal{D}_W)\varphi_n = \lambda_n\varphi_n, \quad \langle \varphi_n^*, \varphi_m\rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t)\langle \mathcal{B}_j, \varphi_n^*\rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j := \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \varphi_j, \quad j = 1,\ldots,m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j) \geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1,\ldots,m$$

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\varphi_n(x), \quad (\mathcal{A}+\mathcal{D}_W)\varphi_n = \lambda_n\varphi_n, \quad \langle \varphi_n^*, \varphi_m\rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t)\langle \mathcal{B}_j, \varphi_n^*\rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j := \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \varphi_j, \quad j = 1,\ldots,m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j)\geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1,\ldots,m$$

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\varphi_n(x), \quad (\mathcal{A}+\mathcal{D}_W)\varphi_n = \lambda_n\varphi_n, \quad \langle \varphi_n^*, \varphi_m\rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t)\langle \mathcal{B}_j, \varphi_n^*\rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j := \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \varphi_j, \quad j = 1,\dots,m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j) \geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1,\dots,m$$

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\varphi_n(x), \quad (\mathcal{A}+\mathcal{D}_W)\varphi_n = \lambda_n\varphi_n, \quad \langle \varphi_n^*, \varphi_m\rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t)\langle \mathcal{B}_j, \varphi_n^*\rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j := \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \varphi_j, \quad j = 1,\dots,m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j) \geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1,\dots,m$$

How to choose $\mathcal{B}_j$ appropriately?

- **Eigen-expansion:**

$$y(t,x) = \sum_{n\geq 1} c_n(t)\,\varphi_n(x), \quad (\mathcal{A}+\mathcal{D}_W)\varphi_n = \lambda_n\varphi_n, \quad \langle \varphi_n^*, \varphi_m\rangle = \delta_{nm}$$

- **Mode dynamics:**

$$\dot{c}_n = \lambda_n c_n + \sum_{j=1}^{m} u_j(t)\langle \mathcal{B}_j, \varphi_n^*\rangle$$

- **Design inputs to target unstable modes:** Solve

$$\mathcal{B}_j := \nabla\cdot(\bar{\mu}\nabla\alpha_j) = \varphi_j, \quad j = 1,\dots,m$$

- **Decoupled slow block:** For the m modes with $\Re(\lambda_j) \geq -\delta$, we get

$$\dot{c}_j = \lambda_j c_j + u_j, \quad j = 1,\dots,m$$

$\mathcal{B}_j = \varphi_j \implies$ infinite-dimensional Hautus condition $\implies$ δ-stability of the linearised system

Local Exponential Stabilisation: Closed-Loop View

The equation for $\psi = e^{\delta t}\mathcal{U}y$ is

$$\dot{\psi} = -\mathcal{H}_{\Pi}\psi - \left(\hat{\mathcal{B}}^*\hat{\Pi}\psi\right)\hat{\mathcal{N}}^{\delta}(t)\psi + \hat{\mathcal{W}}^{\delta}(\psi), \quad \psi(0,\cdot) = \mathcal{U}(\mu_0 - \bar{\mu})$$

where $\mathcal{H}_{\Pi} = \mathcal{H} - \delta I + \hat{\mathcal{B}}\hat{\mathcal{B}}^*\hat{\Pi}$.

- **Linear decay:** $\mathcal{H}_{\Pi}$ generates exponential decay with rate δ
- **Nonlinear remainder:** the term $\hat{\mathcal{W}}^{\delta}(\psi)$ allows a Lipschitz-type estimate
- **Contraction argument:** for $\|\psi_0\|$ sufficiently small, the full closed-loop system is a **contraction** in a suitable function space[9]. Therefore,

$$\|\psi(t)\|_{L^2(\Omega)} \leq C\|\psi_0\|_{L^2(\Omega)} \implies \|y(t)\|_{L^2(\Omega,\bar{\mu}^{-1})} \leq Ce^{-\delta t}\|\mu_0 - \bar{\mu}\|_{L^2(\Omega,\bar{\mu}^{-1})}.$$

Local Exponential Stabilisation: Closed-Loop View

The equation for $\psi = e^{\delta t}\mathcal{U}y$ is

$$\dot{\psi} = -\mathcal{H}_{\Pi}\psi - (\hat{\mathcal{B}}^*\hat{\Pi}\psi)\hat{\mathcal{N}}^{\delta}(t)\psi + \hat{\mathcal{W}}^{\delta}(\psi), \quad \psi(0,\cdot) = \mathcal{U}(\mu_0 - \bar{\mu})$$

where $\mathcal{H}_{\Pi} = \mathcal{H} - \delta I + \hat{\mathcal{B}}\hat{\mathcal{B}}^*\hat{\Pi}$.

- **Linear decay:** $\mathcal{H}_{\Pi}$ generates exponential decay with rate δ
- **Nonlinear remainder:** the term $\hat{\mathcal{W}}^{\delta}(\psi)$ allows a Lipschitz-type estimate
- **Contraction argument:** for $\|\psi_0\|$ sufficiently small, the full closed-loop system is a **contraction** in a suitable function space[9]. Therefore,

$$\|\psi(t)\|_{L^2(\Omega)} \leq C\,\|\psi_0\|_{L^2(\Omega)} \implies \|y(t)\|_{L^2(\Omega,\bar{\mu}^{-1})} \leq Ce^{-\delta t}\|\mu_0 - \bar{\mu}\|_{L^2(\Omega,\bar{\mu}^{-1})}.$$

Local Exponential Stabilisation: Closed-Loop View

The equation for $\psi = e^{\delta t}\mathcal{U}y$ is

$$\dot{\psi} = -\mathcal{H}_{\Pi}\psi - \big(\hat{\mathcal{B}}^*\hat{\Pi}\psi\big)\hat{\mathcal{N}}^{\delta}(t)\psi + \hat{\mathcal{W}}^{\delta}(\psi), \quad \psi(0,\cdot) = \mathcal{U}(\mu_0 - \bar{\mu})$$

where $\mathcal{H}_{\Pi} = \mathcal{H} - \delta I + \hat{\mathcal{B}}\hat{\mathcal{B}}^*\hat{\Pi}$.

- **Linear decay:** $\mathcal{H}_{\Pi}$ generates exponential decay with rate δ
- **Nonlinear remainder:** the term $\hat{\mathcal{W}}^{\delta}(\psi)$ allows a Lipschitz-type estimate
- **Contraction argument:** for $\|\psi_0\|$ sufficiently small, the full closed-loop system is a **contraction** in a suitable function space[9]. Therefore,

$$\|\psi(t)\|_{L^2(\Omega)} \le C\,\|\psi_0\|_{L^2(\Omega)} \implies \|y(t)\|_{L^2(\Omega,\bar{\mu}^{-1})} \le Ce^{-\delta t}\|\mu_0 - \bar{\mu}\|_{L^2(\Omega,\bar{\mu}^{-1})}.$$

Local Exponential Stabilisation: Closed-Loop View

The equation for $\psi = \mathrm{e}^{\delta \mathrm{t}}\mathcal{U}\mathrm{y}$ is

$$\dot{\psi} = -\mathcal{H}_\Pi \psi - \left(\hat{\mathcal{B}}^* \hat{\Pi} \psi\right) \hat{\mathcal{N}}^\delta(\mathrm{t})\psi + \hat{\mathcal{W}}^\delta(\psi), \quad \psi(0,\cdot) = \mathcal{U}(\mu_0 - \bar{\mu})$$

where $\mathcal{H}_\Pi = \mathcal{H} - \delta \mathrm{I} + \hat{\mathcal{B}}\hat{\mathcal{B}}^* \hat{\Pi}$.

- **Linear decay:** $\mathcal{H}_\Pi$ generates exponential decay with rate δ
- **Nonlinear remainder:** the term $\hat{\mathcal{W}}^\delta(\psi)$ allows a Lipschitz-type estimate
- **Contraction argument:** for $\|\psi_0\|$ sufficiently small, the full closed-loop system is a **contraction** in a suitable function space[9]. Therefore,

$$\|\psi(\mathrm{t})\|_{\mathrm{L}^2(\Omega)} \leq \mathrm{C}\, \|\psi_0\|_{\mathrm{L}^2(\Omega)} \implies \|\mathrm{y}(\mathrm{t})\|_{\mathrm{L}^2(\Omega,\bar{\mu}^{-1})} \leq \mathrm{C}\mathrm{e}^{-\delta \mathrm{t}} \|\mu_0 - \bar{\mu}\|_{\mathrm{L}^2(\Omega,\bar{\mu}^{-1})}.$$

[9] D. Kalise, L. M, G. Pavliotis. Linearization-Based Feedback Stabilization of McKean-Vlasov PDEs, arXiv. 2025.

Numerical implementation

Work on the periodic domain $\Omega = \mathbb{T}^d$

- **Fourier expansion:** for $k \in \mathbb{Z}^d$,

$$y(x,t) \approx \frac{1}{\sqrt{2\pi}} \sum_{|k|\le L} \hat{y}_k(t) e^{ik\cdot x}$$

- **Galerkin projection:** project the full equation onto the span of $\left\{ \frac{1}{\sqrt{2\pi}} e^{ik\cdot x} \right\}_{|k|\le L}$. The matrix representation is

$$\dot{\hat{y}} = A\hat{y} + \sum_{j=1}^{m} B_j u_j, \qquad J = \int_0^\infty e^{2\delta t} (\hat{y}^\top M \hat{y} + \|u\|^2)\, dt$$

- **Feedback computation:** solve the discrete Riccati equation

$$(A^\top + \delta I)\Pi_L + \Pi_L (A + \delta I) - \Pi_L B B^\top \Pi_L + M = 0, \quad u = -B^\top \Pi_L \hat{y}$$

Noisy Kuramoto under feedback control

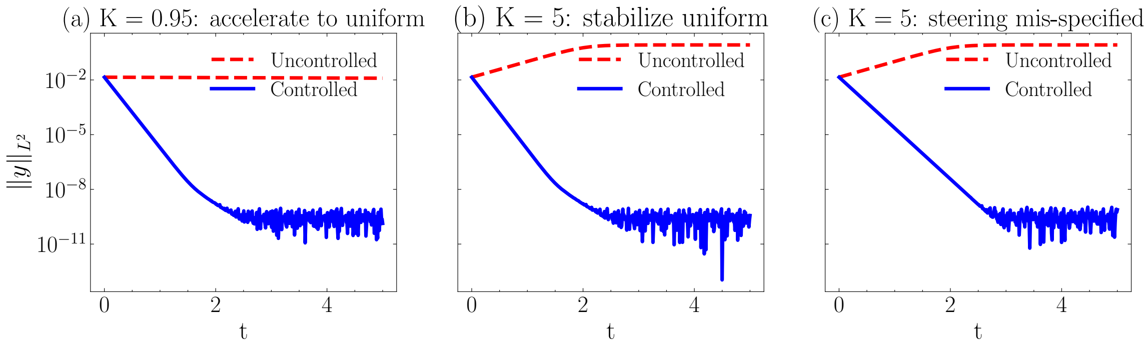


Figure: Set $V(x) = 0$, $W(x) = -K\cos(x)$ and $\sigma = 0.5$.

Kuramoto model + Potential V

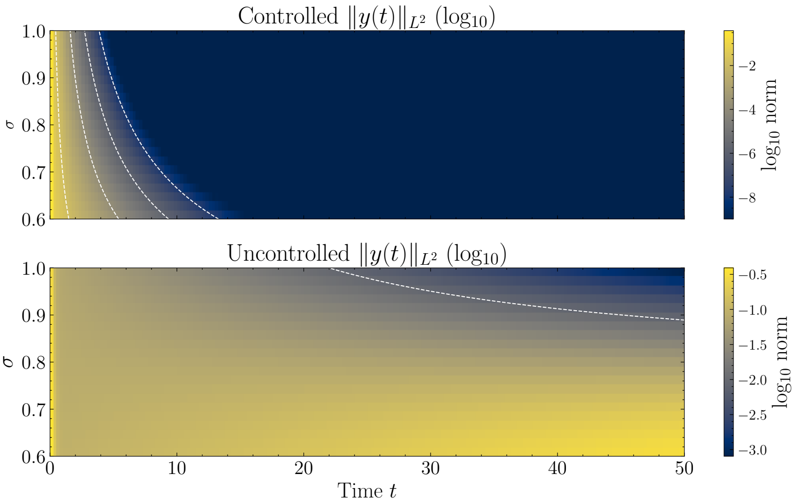


Figure: Set $V(x) = 0.05\cos(x)$ and $W(x) = -\cos(x)$

Outlook

Conclusions

- Deterministic feedback accelerates convergence to steady states.
- Riccati-based law yields local, rate-guaranteed stabilization.

Future Work

- Develop a global Lyapunov feedback law for full nonlinear PDE.
- Analyze robustness under model uncertainties in V and W.
- Numerics to higher-dimensional domains.
- Extend to kinetic equations: hypoelliptic analysis

IMPERIAL

Thank you for the attention! Questions, suggestions?

Operator-Theoretic Reformulation

Weighted spaces and linearisation

- **Why work in weighted spaces?**
  - This is the natural energy space for the linearised and localised operator

$$(\mathcal{A} + \mathcal{D}_{W,1})y := \nabla \cdot (\sigma \nabla y + y \nabla V) + \nabla \cdot (y \nabla W * \bar{\mu})$$

  - It ensures that the operator $\mathcal{A} + \mathcal{D}_{W,1}$ is self-adjoint.
  - Inner product structure simplifies spectral and stability analysis.
  - By a unitary transformation, we convert $\mathcal{A} + \mathcal{D}_W$ into a Schrödinger operator plus a compact operator.

- **Two linearisation strategies:**

  One can adopt the full linearisation $\mathcal{D}_W$, which captures all nonlocal effects, or the simplified local form $\mathcal{D}_{W,1}$, which is easier to handle computationally. Both yield implementable schemes.

$$\overbrace{\sigma \Delta y + \nabla \cdot (y \nabla (V + W * \bar{\mu}))}^{\text{Linear Fokker-Planck operator}} \; + \; \nabla \cdot (\bar{\mu} \nabla W * y)$$