Lista de Exercícios 3

Estatística Bayesiana

4. Let P^* be a given probability distribution on the set S of positive integers $\{1, 2, \ldots\}$ such that each integer in S is assigned a positive probability. Let $S_1 = \{1, 3, 5, \ldots\}$, and let $S_2 = \{2, 4, 6, \ldots\}$. Hence, every subset A of S can be expressed in the form $A = (AS_1) \cup (AS_2)$. Suppose that a relation $\precsim$ is defined between subsets of S as follows: If A and B are any two subsets of S, then $A \precsim B$ if either $P^*(AS_1) < P^*(BS_1)$ or $P^*(AS_1) = P^*(BS_1)$ and $P^*(AS_2) \leq P^*(BS_2)$. Show that the relation $\precsim$ satisfies Assumptions SP_1 to SP_3 but not Assumption SP_4.

4. (SP_1). Tome A, B eventos de S. Defina $p_i^j = P^*(jS_i)$ para $j \in \{A, B\}$ e $i \in \{1, 2\}$.

(i) $p_1^A < p_1^B$ implica $A \precsim B$. Em particular, é impossível que $B \precsim A$, pois $p_1^B \not< p_1^A$ e $p_1^B \neq p_1^A$, isto é $A \prec B$. Concluo que $p_1^A < p_1^B \Rightarrow A \prec B$.

(ii) $p_1^A > p_1^B$. Por analogia ao item (i), $B \prec A$.

(iii) $p_1^A = p_1^B$. Se $p_2^A < p_2^B$, então $A \prec B$, pois é impossível que $B \precsim A$. Se $p_2^B < p_2^A$, então $B \prec A$. Por fim, se $p_2^A = p_2^B$, vale que $A \sim B$.

Pela tricotomia nos reais, vale (SP_1)

(SP_2) Tome A_1, A_2, B_1, B_2 eventos em S com $A_1A_2 = B_1B_2 = \emptyset$ e $A_i \precsim B_i$, $i = 1, 2$. Assim

$$
\begin{aligned}
P^*((A_1 \cup A_2) S_1) &= P^*(A_1 S_1) + P^*(A_2 S_1) \\
(*) &\leq P^*(B_1 S_1) + P^*(B_2 S_1) \\
&= P^*((B_1 \cup B_2) S_1),
\end{aligned}
$$

logo $A_1 \cup A_2 \precsim B_1 \cup B_2$. Sem perda de generalidade, suponha que $A_1 \prec B_1$, i.e., $P^*(A_1 S_1) < P^*(B_1 S_1)$ ou $P^*(AS_1) = P^*(B_1 S_1)$

e $P^*(A, S_2) < P^*(B, S_2)$. Nesse caso é claro que (*) se torna uma desigualdade estrita e está provado.

(SP3) $P^*(\emptyset S_1) = 0 \leq P^*(A S_1) \Rightarrow \emptyset \precsim A$. Além do mais $P^*(S S_1) + P^*(S S_2) = P^*(S_1) + P^*(S_2) = 1$, logo $P^*(S_i) > P(\emptyset S_i)$ e $\emptyset \prec S$.

(SP4) Vamos mostrar que não satisfaz (SP4).

vale pois $P^*(\{i\}) > 0$

Defina $A_i = \{i, i+1, \dots\}$ e suponha $P^*(A_i S_1) > 0$ para todo $i \in \mathbb{N}$. Defina $B = S_2$. Assim

$$0 = P^*(B S_1) < P^*(A_i S_1), \quad \forall i \in \mathbb{N}.$$

Como $A = \bigcap_{i \in \mathbb{N}} A_i = \emptyset$, temos que

$$0 = P^*(B S_1) = P^*(A S_1),$$

mas $P^*(B S_2) > P^*(A S_2) = 0$, isto é, $B \succ A$, o que contradiz (SP4). Note que assumimos que $P^*(S_2) > 0$.

9. Think of a fixed site outside the building in which you are at this moment. Let X be the temperature at that site at noon tomorrow. Choose a number x_1 such that:

(a) $P(X < x_1) = P(X > x_1) = \frac{1}{2}$.

Next, choose a number x_2 such that:

(b) $P(X < x_2) = P(x_2 < X < x_1) = \frac{1}{4}$.

Finally, choose numbers x_3 and x_4 $(x_3 < x_1 < x_4)$ such that:

(c) $P(X < x_3) + P(X > x_4) = P(x_3 < X < x_1) = P(x_1 < X < x_4) = \frac{1}{3}$.

Using the values of x_1 and x_2 that you have chosen and tables of the standard normal distribution, find the unique normal distribution for X that satisfies the relations in parts a and b. Assuming that X has this normal distribution, find from the tables the values which x_3 and x_4 must have in order to satisfy the relation in part c and compare them with the values that you have chosen. Decide whether or not your distribution for X can be represented approximately by a normal distribution.

9. (a) $x_1 = 25$

(b) $x_2 = 20$

(c) $x_3 = 18$, $x_4 = 32$

Suponha $X \sim N(\mu, \sigma^2)$. Como $E[X] = \mu$ e a distribuição é simétrica, a mediana x_1 coincide com a média, i.e., $\mu = 25$.

Note que $\frac{X-25}{\sigma} \sim N(0,1)$, logo

$$\mathbb{P}\left(\frac{X-25}{\sigma} \leq t\right) = \frac{1}{4} \Rightarrow t \approx -0.674,$$

logo $t = \frac{x_2 - 25}{\sigma} = \frac{-5}{\sigma} \Rightarrow \sigma \cong 7.413$

Nesse caso $\mathbb{P}(X < x_3) = \frac{1}{6} \Rightarrow x_3 \approx 17{,}83$

$$\mathbb{P}(X > x_4) = \frac{1}{6} \Rightarrow x_4 \approx 32{,}17,$$

que é similar aos valores adotados.