Lista de exercícios 7

Estatística Bayesiana

5.2 Consider $x \sim \mathcal{N}(\theta, 1)$. The hypothesis to test is $H_0 : |\theta| \leq c$ versus $H_1 : |\theta| > c$ when $\pi(\theta) = 1$.

a. Give the graph of the maximal probability of H_0 as a function of c.

b. Determine the values of c for which this maximum is 0.95 and the Bayes factor is 1. Are these values actually appealing?

a) $$\pi(|\theta| \leq c \mid x) = \frac{\int_{-c}^{c} \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2} d\theta}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2} d\theta}$$

$$= \Phi(c-x) - \Phi(-c-x)$$

$$= 1 - \Phi(x-c) - 1 + \Phi(x+c)$$

$$= \Phi(x+c) - \Phi(x-c)$$

Defina $g(c) := \max_x \pi(|\theta| \leq c \mid x)$.

$$\frac{\partial}{\partial x} \pi(|\theta| \leq c \mid x) = \varphi(x+c) - \varphi(x-c)$$
$$= \frac{1}{\sqrt{2\pi}} \left(e^{-(x+c)^2/2} - e^{-(x-c)^2/2} \right) = 0$$

$$\Leftrightarrow e^{-(x+c)^2/2} = e^{-(x-c)^2/2} \Leftrightarrow (x+c)^2 = (x-c)^2$$
$$\Leftrightarrow |x+c| = |x-c|.$$

Se $c > 0$, a única solução é $x + c = c - x \Rightarrow x = 0$. Note que $$\frac{\partial^2}{\partial x^2} \pi(|\theta| \leq c \mid 0) = \frac{1}{\sqrt{2\pi}} \left(-2c e^{-c^2/2} - 2c e^{-c^2/2} \right) < 0$$

$$\Rightarrow g(c) = \Phi(c) - \Phi(-c) = 2\Phi(c) - 1$$

b) $g(c) = 0.95 \Rightarrow c = \Phi^{-1}(1.95/2)$. O fator de Bayes depende de x.

5.6 When $x \sim \mathcal{N}(\theta, 1)$ and $\theta \sim \mathcal{N}(0, \sigma^2)$, compare the Bayesian answers for the two testing problems

$$H_0^1: \quad \theta = 0 \text{ versus } H_1^1: \ \theta \neq 0,$$
$$H_0^2: \quad |\theta| \leq \epsilon \text{ versus } H_1^2: \ |\theta| > \epsilon,$$

when ϵ and σ vary.

Já sabemos que $\theta | x \sim N(x\tau, \tau)$, $\tau = (1+\sigma^{-2})^{-1}$

No caso de H_0^1:

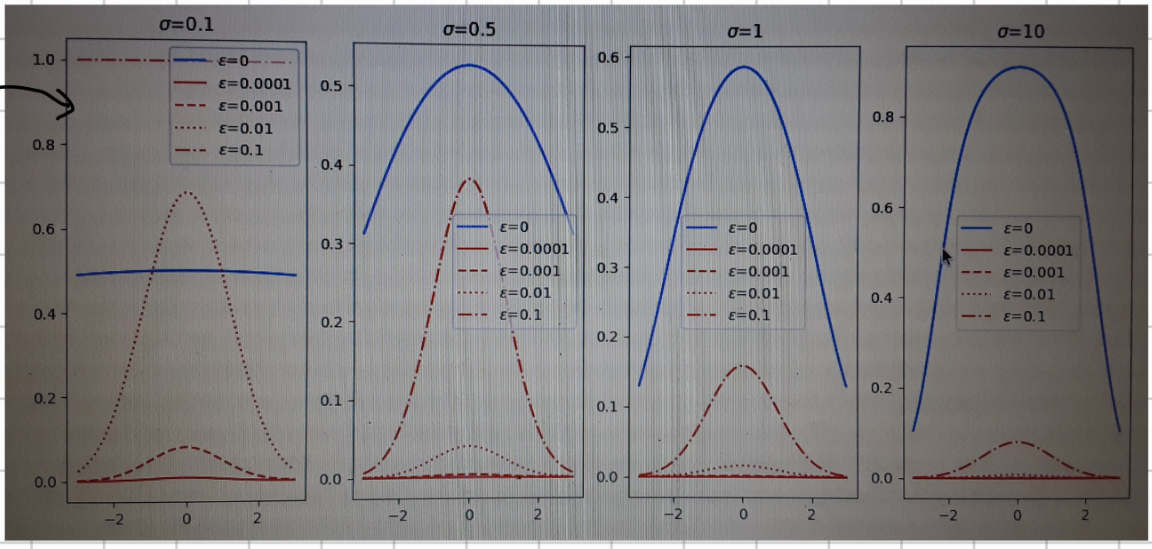
$$m_1(x) = \frac{(2\pi)^{-1/2} \exp\left\{-\frac{1}{2}(x-\theta)^2\right\} (2\pi)^{-1/2} \sigma^{-1} \exp\left\{-\frac{1}{2\sigma^2}\theta^2\right\}}{(2\pi)^{-1/2}\sqrt{1+\sigma^{-2}} \exp\left\{-\frac{1}{2\tau}(\theta - x\tau)^2\right\}}$$
$$= (2\pi)^{-1/2}(1+\sigma^2)^{-1/2} \exp\left\{-\frac{(1-\tau)}{2}x^2\right\},$$
$$\Rightarrow x \sim N(0, 1+\sigma^2).$$

Logo $$\pi(\theta = 0 | x) = \left[1 + \frac{(1-\rho_0)}{\rho_0\sqrt{1+\sigma^2}} \exp\left\{\frac{x^2}{2(1+\sigma^{-2})}\right\}\right]^{-1}$$

No caso de H_0^2:

$$\pi(|\theta| \leq \varepsilon | x) = \Phi\left(\varepsilon(1+\sigma^{-2}) - x\right) - \Phi\left(-\varepsilon(1+\sigma^{-2}) - x\right)$$
$$= \Phi\left(x + \varepsilon(1+\sigma^{-2})\right) - \Phi\left(x - \varepsilon(1+\sigma^{-2})\right)$$

$\rho_0 = 1/2$



5.12 In a normal setting, determine whether there exists a normalization problem associated with noninformative prior distributions for tests of one-sided hypotheses such as

$$H_0: \theta \in [0,1] \quad \text{versus} \quad H_1: \theta > 1.$$

Replace 1 by ϵ and consider the evolution of the optimal answer as ϵ goes to 0.

Seja $x \sim N(\theta, \sigma^2)$ com σ conhecido. Defina

$$\pi(\theta) \propto 1 \cdot \mathbb{1}\{\theta \geq 0\}$$

Assim

$$\pi(\theta|x) \propto \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}\mathbb{1}\{\theta \geq 0\}.$$

e $\int_0^{+\infty} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\} = \sqrt{2\pi}\,\sigma\,\Phi(x/\sigma)$, isto é,

$$\pi(\theta|x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta-x)^2}{2\sigma^2}} \cdot \frac{\mathbb{1}\{\theta \geq 0\}}{\Phi(x/\sigma)}$$

$$\theta|x \sim N_+(x, \sigma^2)$$

$$\begin{aligned}
\pi(\theta \in [0,\varepsilon]\,|x) &= \int_0^{\varepsilon} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\theta-x)^2}{2\sigma^2}} \frac{1}{\Phi(x/\sigma)}\, d\theta \\
&= \frac{\Phi\left(\frac{\varepsilon-x}{\sigma}\right) - \Phi\left(\frac{-x}{\sigma}\right)}{\Phi(x/\sigma)} \\
&= \frac{\Phi\left(\frac{\varepsilon-x}{\sigma}\right) - 1}{\Phi(x/\sigma)} + 1 \\
&= 1 - \frac{\Phi\left(\frac{x-\varepsilon}{\sigma}\right)}{\Phi(x/\sigma)}
\end{aligned}$$

Com isso, se observamos $x \to -\infty$, $\pi(H_0|x) \to 1$, e, portanto, não temos um limite superior não trivial nesse caso. Mesmo quando $x > 0$, o upper bound de 0.7 é razoável.

Agora considere $\pi(\theta) \propto c_0 \mathbb{1}\{\theta \in [0, \varepsilon]\} + c_1 \mathbb{1}\{\theta > \varepsilon\}$.

Com isso, temos que

$$\pi(\theta \in [0,\varepsilon] \mid x) = \frac{c_0 \displaystyle\int_{\cancel{0}\; -x/\sigma}^{\cancel{\varepsilon}\; \frac{\varepsilon - x}{\sigma}} \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{\cancel{(x-\theta)^2}\; z^2}{2\cancel{\sigma^2}}} \, d\cancel{\theta}z}{c_0 \displaystyle\int_{\cancel{0}\; -x/\sigma}^{\cancel{\varepsilon}\; \frac{\varepsilon - x}{\sigma}} \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{\cancel{(x-\theta)^2}\; z^2}{2\cancel{\sigma^2}}} \, d\cancel{\theta}z + c_1 \displaystyle\int_{\cancel{\varepsilon}\; \frac{\varepsilon - x}{\sigma}}^{+\infty} \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{\cancel{(x-\theta)^2}\; z^2}{2\cancel{\sigma^2}}} \, d\cancel{\theta}z}$$

$$= \left[1 + \frac{c_1}{c_0} \left(\frac{1 - \Phi\left(\frac{\varepsilon - x}{\sigma}\right)}{\Phi\left(\frac{\varepsilon - x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right)} \right) \right]^{-1}$$

$$= \left[1 + \frac{c_1}{c_0} \left(\frac{\Phi\left(\frac{x - \varepsilon}{\sigma}\right)}{\Phi\left(\frac{x}{\sigma}\right) - \Phi\left(\frac{x-\varepsilon}{\sigma}\right)} \right) \right]^{-1}$$

$$= \left[1 + \frac{c_1}{c_0} \left(\frac{\Phi(x/\sigma)}{\Phi\left(\frac{x - \varepsilon}{\sigma}\right)} - 1 \right)^{-1} \right]^{-1},$$

que depende de c_1/c_0. Quando $\varepsilon \to 0$, acontece que $\pi(\theta \in [0,\varepsilon] \mid x) \to 0$, como esperado.

Para encerrar a análise, considere $\theta \sim N_+(0, \tau^2)$. Assim

$$\theta \mid x \sim N_+\left(x(\sigma^2/\tau^2 + 1)^{-1}, (\tau^{-2} + \sigma^{-2})^{-1} \right)$$

$$\pi(\theta \in [0, \varepsilon] \mid x) = 2 \int_0^{\varepsilon} \frac{\sqrt{(\tau^{-2} + \sigma^{-2})}}{\sqrt{2\pi}} e^{-\frac{\tau^{-2}+\sigma^{-2}}{2}\left(\theta - x(\sigma^2/\tau^2 + 1)^{-1}\right)^2} d\theta$$

$$= 2 \left[\Phi\left(\frac{\varepsilon - x(\sigma^2/\tau^2 + 1)^{-1}}{(\tau^{-2} + \sigma^{-2})^{-1/2}} \right) - \Phi\left(\frac{-x(\sigma^2/\tau^2 + 1)^{-1}}{(\tau^{-2} + \sigma^{-2})^{-1/2}} \right) \right]$$

Quando $\tau \to +\infty$, isso converge a $2\left(\Phi\left(\frac{\varepsilon - x}{\sigma}\right) - \Phi\left(-\frac{x}{\sigma}\right)\right)$. Agora, se $\varepsilon \to 0$, $\varepsilon(\tau^{-2} + \sigma^{-2})^{1/2} \longrightarrow 0$ e

$$\pi(\theta \in [0, \varepsilon] \mid x) \to 0.$$

5.42. $x \sim P(\lambda)$, $\lambda \sim G(\delta, \beta) \Rightarrow \lambda | x \sim G(\delta + x, \beta + 1)$

Considerando o intervalo de credibilidade

$$\left[G^{-1}(\alpha/2), G^{-1}(1 - \alpha/2) \right]$$

A evolução parece ser linear e o aumento do tamanho do intervalo não é tão grande também. No caso em que $\delta = \beta = 0$, o intervalo é o mais próximo de zero.