Lista de exercícios 8

Estatística Bayesiana

7.1 The *deviance* associated with a model is simply the log-likelihood taken at the maximum likelihood estimator (McCullagh and Nelder (1989)). In the setting of Example 7.1.1, compute the maximum likelihood estimator's $\hat{\lambda}$ and $(\hat{m}, \hat{p})$ and compare both deviances.

Se $x \sim f(x|\theta)$ e $\hat{\theta}(x) = \operatorname{argmax}_\theta f(x|\theta)$, então o desvio

$$D(x) = \log f(x|\hat{\theta}(x)).$$

Considere o exemplo 7.1.1.

$$\mathcal{M}_1 : N \sim \operatorname{Poi}(\lambda) \Rightarrow f_1(N|\lambda) = \frac{1}{N!} \lambda^N e^{-\lambda}.$$

Assim $l_1(\lambda|N) = N \ln \lambda - \lambda - \log N!$ é a log-verossimilhança. Fazendo

$$0 = \frac{\partial}{\partial \lambda} l_1(\lambda|N) = \frac{N}{\lambda} - 1 \Rightarrow \lambda = N$$

e, como

$$\frac{\partial^2}{\partial \lambda^2} l_1(N|N) = -\frac{N}{N^2} = -\frac{1}{N} < 0$$

temos que $\hat{\lambda}(N) = N$ é MLE para λ em $\mathcal{M}_1$. Assim

$$\begin{aligned} D_1(N) &= N(\ln N - 1) - \log N! \\ &= N \ln(N/e) - \log N! \\ &= \textstyle\sum_{i=1}^{N} \ln(N/ie) \end{aligned}$$

Agora $\mathcal{M}_2 : N \sim \operatorname{Neg}(m,p) \Rightarrow f_2(N|m,p) = \binom{m+N-1}{N} p^m (1-p)^N.$

Assim $l_2(m,p|N) = \log \binom{m+N-1}{N} + m \log p + N \log(1-p)$ é a log-verossimilhança. Observe que

$$0 = \frac{\partial}{\partial p} l_2 = \frac{m}{p} - \frac{N}{1-p} \Rightarrow p = \frac{m}{m+N} \quad \text{é o único}$$

ponto crítico da segunda componente. Além disso

$$\frac{\partial^2}{\partial p^2} l_2 = -\frac{m}{p^2} - \frac{N}{(1-p)^2} < 0.$$

Assim, a log-verossimilhança em $\hat{p}$ se torna

$$l_2(N|m) = \log \binom{m+N-1}{N} + m\log m + N\log N - (m+N)\log(m+N),$$

em que queremos maximizar em $m \in \mathbb{N}$. Continuamente,

$$l_2(N|m) = \log\Gamma(m+N) - \log\Gamma(m) + m\log m - (m+N)\log(m+N) + K_N$$

cuja derivada é

$$\frac{\partial}{\partial m} l_2(N|m) = [\psi(m+N) - \log(m+N)] - [\psi(m) - \log(m)]$$

$$> 0,$$

pois a função $\psi(\cdot) - \log(\cdot)$ é crescente. Assim, não existe MLE nesse caso. Em particular,

$$D_2(N) = \lim_{m\to\infty} l_2(N|m)$$

$$= \log\left\{\lim_{m\to\infty} \frac{(m+N-1)!}{(m+N)^{m+N}} \Big/ \frac{(m-1)!}{m^m}\right\} + K_N$$

$$= \log\left\{\lim_{m\to\infty} \prod_{i=1}^{N} \frac{(m+N-i)}{(m+N)} \cdot \left(\frac{m+N}{m}\right)^{-m}\right\} + K_N$$

$$= \log\left\{\prod_{i=1}^{N} \lim_{m\to\infty}\left(1 - \frac{i}{m+N}\right) \cdot \lim_{m\to\infty}\left(\left(1+\frac{1}{m/N}\right)^{-\frac{m}{N}}\right)^{N}\right\} + K_N$$

$$= \log\{e^{-N}\} + K_N = -N + N\log N - \log N!$$

$$= D_1(N).$$

Concluímos que $D_1(N) > D_2(N)$, $\forall N \in \mathbb{N}$.

7.5 In the setting of Example 7.2.1, assume T_t is distributed from a uniform $\mathcal{U}_{[0,\bar{T}]}$ distribution, and that $\beta_{21} \sim \mathcal{N}(0, \tau^2)$.

a. Compute the marginal model of y_{it} by integrating out the term $\beta_{21} T_t$ in $\mathcal{M}_2$.

b. Deduce the prior distribution on the parameters of $\mathcal{M}_1$ if $\mathcal{M}_2$ is the true model and $(\beta_{20}, b_{2i}, \sigma_2) \sim \pi(\beta_{20}, b_{2i}, \sigma_2)$.

$$\mathcal{M}_1: \; y_{it} \sim \mathcal{N}(\beta_{10} + b_{1i}, \sigma_1^2)$$
$$\mathcal{M}_2: \; y_{it} \sim \mathcal{N}(\beta_{20} + b_{2i} + \beta_{21} T_t, \sigma_2^2)$$

Assuma $T_t \sim \mathcal{U}[0, \bar{T}]$ e $\beta_{21} \sim \mathcal{N}(0, \tau^2)$. independente

a) $$f(y_{it} \mid \beta_{20}, b_{2i}, \sigma_2^2) = \int_0^{\bar{T}} \int_{-\infty}^{\infty} f(y_{it} \mid \beta_{20}, b_{2i}, \sigma_2^2, \beta_{21}, T_t) \times \pi(\beta_{21}, T_t \mid \beta_{20}, b_{21}, \sigma_2^2)\, d\beta\, dT$$

$y_{it} - \beta_{20} - b_{2i}$

$$= \int_0^{\bar{T}} \frac{1}{\bar{T}} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_2\tau} \exp\left\{ -\frac{(\overset{=}{y} - xt)^2}{2\sigma_2^2} - \frac{x^2}{2\tau^2} \right\} dx\, dt$$

$$= \int_0^{\bar{T}} \frac{\exp\left\{ -\frac{1}{2} \frac{y^2}{t^2\tau^2 + \sigma_2^2} \right\}}{\sqrt{2\pi(t^2\tau^2 + \sigma_2^2)} \cdot \bar{T}}\, dt,$$

que pode ser computada numericamente.

7.12 Show that, for the comparison of two linear models $\mathcal{M}_1$ and $\mathcal{M}_2$ with k_1 and k_2 regressors, respectively, and n observations, under the prior $\pi_j(\beta_j) = \sigma_j^{-1-q_j}$ $(j = 1, 2)$, the BIC writes down as

$$B_{12} = \left(\frac{R_2}{R_1}\right)^{n/2} n^{(k_2-k_1)/2},$$

where the R_j's are the residual sums of squares.

Suponha $K_2 > K_1$.

$$L_{j,n}(\beta, \sigma) = \frac{1}{(2\pi)^{n/2}\sigma_j^n} \exp\left\{-\frac{1}{2\sigma_j^2}\sum_{i=1}^{n}(y_i - X_i^j\beta_j)^2\right\}, \text{ em que}$$

$$\hat{\beta}_j = (X^{jT}X)^{-1}X^{jT}y$$
$$\hat{\sigma}_j^2 = n^{-1}R_j$$

Logo $\lambda_n = \left(\frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2}\right)^{n/2} \exp\left\{-\frac{1}{2}\left(\frac{R_1}{\hat{\sigma}_1^2} - \frac{R_2}{\hat{\sigma}_2^2}\right)\right\}$

$$= \left(\frac{R_2}{R_1}\right)^{n/2}$$

Além disso $p_j = K_j - 1$ e, portanto $p_2 - p_1 = K_2 - K_1$.

Concluo que $S = -\log\left(\frac{R_2}{R_1}\right)^{n/2} - \frac{(K_2-K_1)}{2}\log(n)$

$\Rightarrow BIC = e^{-S} = \left(\frac{R_2}{R_1}\right)^{n/2} n^{(K_2-K_1)/2}$.

Obs.: Na verdade, $BIC = S$, mas nesse exercício, a preocupação dele é outra.

7.20 Given two densities $\pi_1(\theta) = c_1\tilde{\pi}_1(\theta)$ and $\pi_2(\theta) = c_2\tilde{\pi}_2(\theta)$ on the same parameter space Θ,

a. For an arbitrary function h, express $\mathbb{E}^{\pi_2}[h(\theta)\tilde{\pi}_1(\theta|x)]$ as an integral in terms of π_1 and π_2.

b. Deduce the equality (7.3.4).

a) $$E^{\pi_2}[h(\theta)\tilde{\pi}_1(\theta|x)] = \int_\Theta h(\theta)\tilde{\pi}_1(\theta|x)\,\pi_2(\theta|x)\,d\theta$$

$$= \frac{\int_\Theta h(\theta) f_1(x|\theta)\pi_1(\theta)\, f_2(x|\theta)\pi_2(\theta)\,d\theta}{c_1\, m_1(x)\, m_2(x)}$$

b) $$\frac{E^{\pi_2}[h(\theta)\tilde{\pi}_1(\theta|x)]}{E^{\pi_1}[h(\theta)\tilde{\pi}_2(\theta|x)]} = \frac{\dfrac{\int_\Theta h(\theta) f_1(x|\theta)\pi_1(\theta)\, f_2(x|\theta)\pi_2(\theta)\,d\theta}{c_1\, m_1(x)\, m_2(x)}}{\dfrac{\int_\Theta h(\theta) f_2(x|\theta)\pi_2(\theta)\, f_1(x|\theta)\pi_1(\theta)\,d\theta}{c_2\, m_1(x)\, m_2(x)}}$$

$$= \frac{c_2}{c_1},$$

igualdade 7.3.4.