34th Brazilian Mathematical Colloquium

Optimal Vaccination Strategies in Interconnected Metropolitan Areas

Lucas M. Moschen and Maria S. Aronna

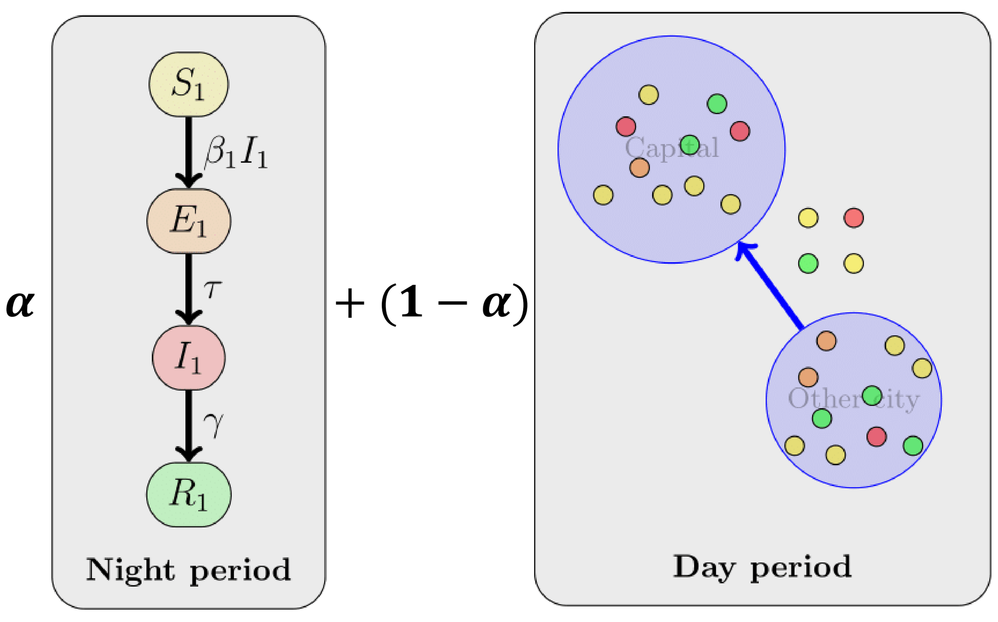
School of Applied Mathematics, Fundação Getulio Vargas, Brazil

Highlights

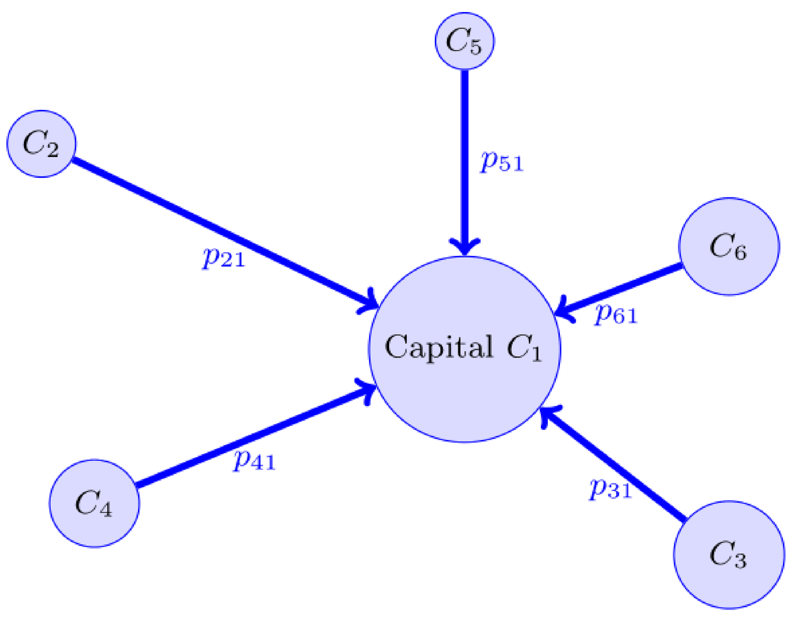
- Developed a mathematical model to track the **spread of an epidemic** in major metropolitan areas, including Rio de Janeiro, Paris, and New York.
- Implemented a centralized vaccination strategy in these regions by solving a **high-dimension optimal control problem with constraints**.
- Conducted numerical experiments that suggest a **higher vaccination rate in the capital** city can be beneficial, depending on the cost of the vaccine.

Modelling strategy

- **Compartmental model** based on [1].
- The metropolitan area is divided in cities, which contains Susceptible, Exposed, Infectious and Recovered individuals as a usual SEIR model.
- A proportion p_{ij} **of the city** i **works in the city** j during the day.
- The individuals spend a proportion α of their day in their home city and $1-\alpha$ **in the capital working**, yielding the following diagram



- We assume a **metropolitan structure:** there is a larger city, the capital, that attracts most of the workers from other cities.
- Then, a parcel p_{k1} **of each city works in the capital**, leading to the following graph:



- The **force of infection** is $\alpha\beta_k I_k$ in the night period and

$$(1-\alpha)(p_{k1}\beta_1 I_1^{eff} + p_{kk}\beta_k I_k^{eff})$$

during the working hours for each non-capital city, where I_k^{eff} is the effective number of infectious individuals in city k during the day.
- The parameter β represents the infection rate, which depends on the **city's density**. The parameter τ is the inverse of **the infectious** period, and γ is the inverse of the **recovery** period.

Basic reproduction number

- We calculate the R_0 using the spectral radius of the **next-generation matrix**, which reflects the rate of new infections and the duration of infectiousness.
- Despite not being able to obtain a closed-form expression, we establish a **general bound**:

$$\min_i v_i \leq R_0 \leq \max_i v_i,$$

in which $v_i = \alpha R_0^i + (1-\alpha)\sum_{j=1}^K p_{ij} R_0^j$.
- Another bound uses the **assumption of a metropolitan area**. Assuming $\alpha \geq 0.5$, we observed numerically that this bound is tighter in 80% of the times.

Optimal control problem

- We include **vaccination** as a control strategy to curb the epidemic: susceptible, exposed and recovered individuals receive a vaccine at **a rate** u_i depending on the city.
- The model aims to minimize a cost functional that balances the **number of vaccinated** and **hospitalized individuals** at the final time T.
- We consider **capacity and logistic restrictions**: a weekly cap of available vaccines and an instantaneous cap of vaccinated individuals.
- The final model is a **mixed control-state** and **pure-state** constrained optimal control problem:

$$\min_u \sum_{i=1}^K c_v n_i V_i(T) + c_h \int_0^T r_h n_i I_i \, dt,$$

$$\text{s.a.} \sum_{j=1}^K V_i(t) n_i \leq D(t), \text{a.e. } t \in [0,T]$$

$$u_i(t)\cdot(S_i(t)+E_i(t)+R_i(t)) \leq D_i, \text{a.e. } t \in [0,T]$$

$$u_i(t) \geq 0, \text{a.e. } t \in [0,T]$$

$$\frac{dS_i}{dt} = -\alpha\beta_i S_i I_i - (1-\alpha)S_i \sum_{j=1}^K \beta_j p_{ij} I_j^{\text{eff}} - u_i S_i$$

$$\frac{dE_i}{dt} = \alpha\beta_i S_i I_i + (1-\alpha)S_i \sum_{j=1}^K \beta_j p_{ij} I_j^{\text{eff}} - \tau E_i - u_i E_i$$

$$\frac{dI_i}{dt} = \tau E_i - \gamma I_i$$

$$\frac{dR_i}{dt} = \gamma I_i - u_i R_i$$

$$\frac{dV_i}{dt} = u_i \cdot (S_i + E_i + R_i)$$

Theoretical results

- We proved **existence of optimal solution** for our problem as an application of Cesari's paper [2].
- We derived the **necessary conditions** based on the work of Boccia, De Pinho and Vinter [3].
- The pure-state constraint poses a challenge due to the **corresponding multiplier being a measure**.
- This allowed us to analyze the general behavior of the solution. We could verify that **the solution attains the bounds**, that is,

$$u_i(S_i + E_i + R_i) \in \{0, D_i\}.$$

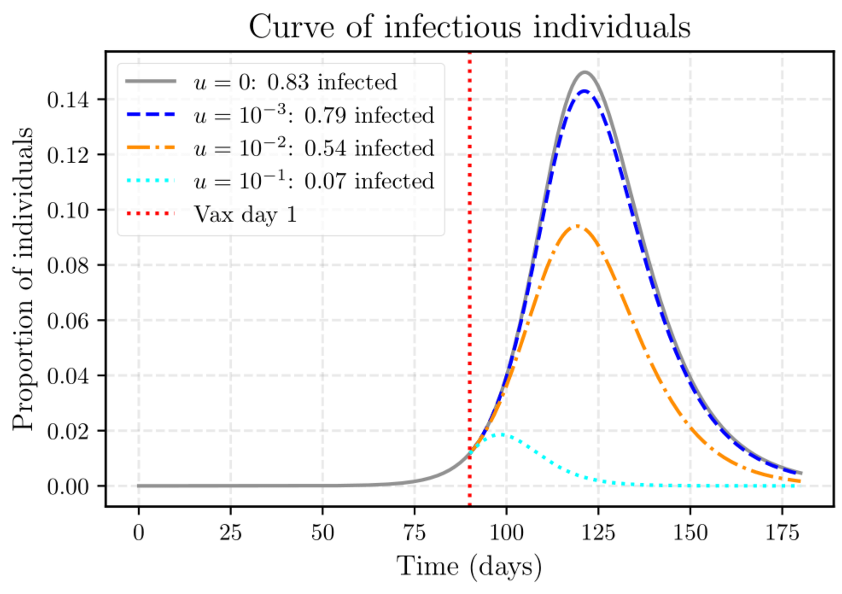
- This result exhibits characteristics like the **Bang Bang solution**, but with variable bounds.

Numerical experiments

- We noticed that R_0 **is most influenced by the value of** β_1, the infection rate of the capital. So, decreasing it is more relevant.
- The parameters α and p_{ij} do not change the behavior of the epidemic but **increase it in more interconnected regions**.
- The metropolitan area assumption resulted in worse upper and lower bounds in only 4% of the time compared to the general bound. We highlight the percentage of times the upper (UB) and lower (LB) bounds were better.

	UB	LB
49%	X	X
46%	X	
1%		X

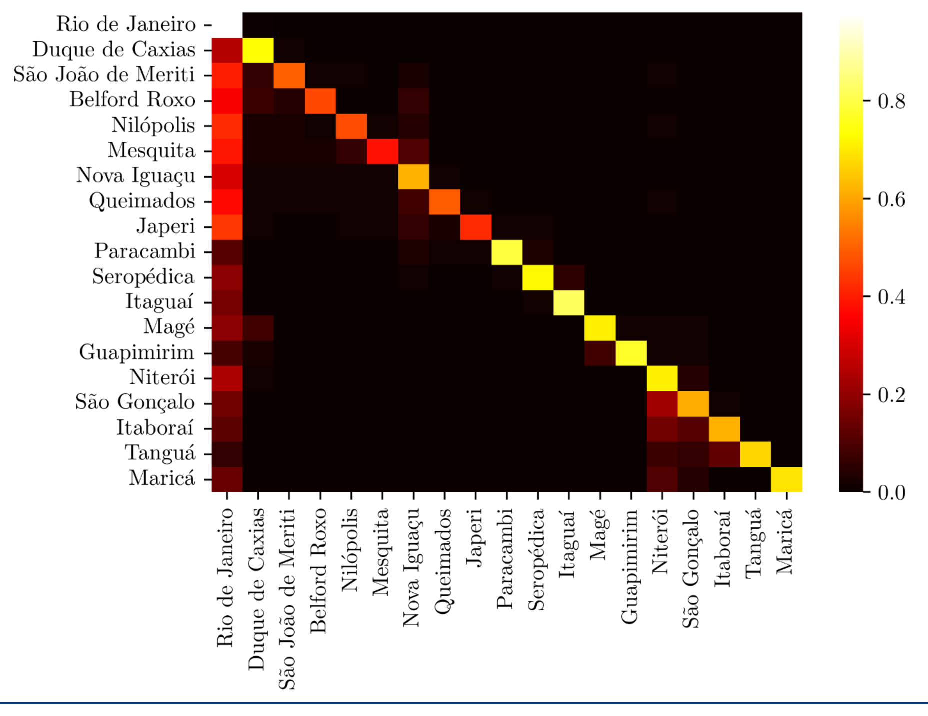
- When including vaccination, we can verify its **relevance in reducing the number of infections** in the metropolitan region:



- We verified that the optimizer preferers to **vaccinate the capital** when the susceptible population still plays a role.

The case of Rio de Janeiro

- The transition matrix of working place in Rio de Janeiro shows our assumption:



Discussion and conclusions

- **Optimal control** is a powerful tool with a robust mathematical foundation that can provide solutions to **real-world problems**.
- A comprehensive understanding of the model can be achieved by studying the problem **theoretically and numerically**.
- There is a **gap in the field** regarding second-order and sufficient conditions for optimality in control-affine problems with constraints. This presents an opportunity for **further research** and development.
- Our study found that a **higher vaccination rate in the capital** accelerate the control of the epidemic. However, the interplay of various factors in this process is complex.

References

1) L. G. Nonato, P. Peixoto, T. Pereira, C. Sagastizábal, and P. J. S. Silva. Robot Dance: A mathematical optimization platform for intervention against COVID-19 in a complex network. EURO Journal on Computational Optimization 10, 2022.
2) L. Cesari. Existence Theorems for Optimal Solutions in Pontryagin and Lagrange Problems. Journal of the SIAM, Control, 1965.
3) A. Boccia, M. D. R. de Pinho, and R. B. Vinter. Optimal Control Problems with Mixed and Pure State Constraints. SIAM Journal on Control and Optimization, 2016.